ON THE TENSOR PRODUCT OF OPERATORS ON HILBERT SPACE

A THESIS
SUBMITTED TO THE COLLEGE OF SCIENCE
UNIVERSITY OF BAGHDAD
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF MASTER OF SCIENCE IN MATHEMATICS

BY
Maysaa Majid Abdul-Munem Altimimmi

May 2005
GLOSSARY OF THE NOTATION AND DEFINITIONS THAT ARE FREQUENTLY USED IN THE THESIS

$\mathbb{C}$: the field of complex numbers
$H$: complex Hilbert space.
$B(H)$: space of all bounded linear operators on $H$.
$I$: identity operator.
$\langle,\rangle$: the inner product.
$S^*$: complex conjugate of a set $S$.
$A^*$: the adjoint of $A$.
$\text{conv}S$: convex hull of a set $S$.
$\sigma(A)$: the spectrum of $A$.
$\sigma_{\text{ap}}(A)$: approximate point spectrum of $A$.
$\sigma_p(A)$: point spectrum of $A$.
$\Gamma(A)$: compression spectrum of $A$.
$A \otimes B$: tensor product of two operators.
$x \otimes y$: tensor product of two vectors.
$H \otimes K$: tensor product of Hilbert spaces.
$x \bar{\otimes} y$: tensor product defined by $(x \bar{\otimes} y)z = \langle z, y \rangle x$ for all $z$.
$r(A)$: spectral radius of $A$.
$W(A)$: numerical range of $A$.
$C(S)$: convex hull of set $S$. 

$\text{conv}S$: convex hull of a set $S$.
$\sigma(A)$: the spectrum of $A$.
$\sigma_{\text{ap}}(A)$: approximate point spectrum of $A$.
$\sigma_p(A)$: point spectrum of $A$.
$\Gamma(A)$: compression spectrum of $A$.
$A \otimes B$: tensor product of two operators.
$x \otimes y$: tensor product of two vectors.
$H \otimes K$: tensor product of Hilbert spaces.
$x \bar{\otimes} y$: tensor product defined by $(x \bar{\otimes} y)z = \langle z, y \rangle x$ for all $z$.
$r(A)$: spectral radius of $A$.
$W(A)$: numerical range of $A$.
ACKNOWLED
THANKS GOOD FOR EVERY THING

Then I am very grateful to my supervisor
Dr. Buthainah Abd Al -Hussun
for her excellent guidance, encouragement, wide opinions
and fruitful discussion.
Special thanks to Dr. Adil G. Naoum for his valuable
assistance during this work .

I am also very grateful to the head and the staff members of
the department of Mathematics and also to the college of
Science of university of Baghdad.
<table>
<thead>
<tr>
<th>Chapter one</th>
<th>some preliminary concepts</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1. Preliminaries.</td>
<td>4</td>
</tr>
<tr>
<td>1.2. Tensor product in vector spaces.</td>
<td>10</td>
</tr>
<tr>
<td>1.3. Tensor product of operators.</td>
<td>15</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Chapter two</th>
<th>some properties of operators that are invariant under tensor product part I</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1. Some classes of operators.</td>
<td>26</td>
</tr>
<tr>
<td>2.2. Tensor product of some classes of operators.</td>
<td>32</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Chapter three</th>
<th>some properties of operators that are invariant under tensor product part II</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.1. Compactness of Tensor product operators.</td>
<td>52</td>
</tr>
<tr>
<td>3.2. The relation between Tensor product and elementary operators.</td>
<td>60</td>
</tr>
<tr>
<td>3.3. Tensor product and strong stability.</td>
<td>66</td>
</tr>
</tbody>
</table>
Abstract

Let $H_1$ and $H_2$ be two vector spaces over a field $K$ the formal linear combination of pairs $(f, g)$ is denoted by $F(H_1, H_2)

F(H_1, H_2) = \left\{ \sum_{j=1}^{n} c_j(f_j, g_j) : c_j \in K, f_j \in H_1, g_j \in H_2, j = 1, K, n \right\}

is a vector space.

Let $N$ be the subspace of $F(H_1, H_2)$ spanned by the elements of the form.

$$\sum_{j=1}^{n} \sum_{k=1}^{m} a_j b_k (f_j, g_k) - 1 \times \left( \sum_{j=1}^{n} a_j f_j, \sum_{k=1}^{m} b_k g_k \right)$$

The quotient space.

$H_1 \otimes H_2 = F(H_1, H_2) / N$ is called the algebraic tensor product.

$H_1 \otimes H_2$ is a pre-Hilbert space with inner product defined by

$$\left\langle \sum_{j=1}^{n} c_j (f_j \otimes g_j), \sum_{k=1}^{m} c'_k (f'_k \otimes g'_k) \right\rangle = \sum_{j=1}^{n} \sum_{k=1}^{m} \overline{c_j} c'_k \langle f_j, f'_k \rangle \langle g_j, g'_k \rangle$$

The completion of $H_1 \otimes H_2$ is called the tensor product of the Hilbert spaces $H_1, H_2$ and denoted by $H_1 \hat{\otimes} H_2$.

Robert, I. showed that every operator $A$ on the Hilbert space $H_1 \hat{\otimes} H_2$ can be written as tensor product of two operators $A_1$ and $A_2$ and denoted by $A_1 \otimes A_2$ is defined by

$$\langle u \otimes w, A(v \otimes z) \rangle = \langle u, A_1 v \rangle \langle w, A_2 z \rangle = \langle u \otimes w, A_1 v \otimes A_2 z \rangle$$

for all $u, v, w, z \in H_2$

In this thesis we study some properties of tensor product of operators defined on $H_1 \hat{\otimes} H_2$ where each $H_1$ and $H_2$ is separable Hilbert space and look for the relation between tensor products of operators with some kind of operators.

Also we study the identification between the elementary operator with $A_1 \otimes A_2^*$ and we can study the property of $A_1 \otimes A_2^*$ by studying the property of $\tau_{A_1, A_2}(\lambda)$

Throughout this thesis we exhibits some known result, with more details, give proofs for other ones, many results are new to the best of our knowledge, in particular those in chapter two.
المستخلص

ل Máyن ظل من فضاء متبعاه معرفه على الحقل $K$ تعرف الطرقية

الخطيمن الأزواج المرتبة $(f,g)$ بالشكل

$$F(H_1,H_2) = \left\{ c_j : c_j \in K, f_j \in H_1, g_j \in H_2, j = 1, K, n \right\}$$

"فضاء خطياً $F(H_1,H_2)$" وليكن فضاء $N$ مولد بالعناصر حافية الشكل

$$\sum_{j=1}^{n} a_j b_k \left( f_j, g_k \right) - 1 \times \left( \sum_{j=1}^{n} a_j f_j, \sum_{k=1}^{m} b_k g_k \right)$$

يعبر بالشكل $N$ بأنه فضاء القسمه على $H_1 \otimes H_2$ بالجديد $H_1 \otimes H_2 = F(H_1,H_2)_N$ بالرمز

بين روابط أن المؤثر $A$ في فضاء هليبري يمكن تحديبه على شكل $H_1 \otimes H_2$ ويتبع بالشكل $A_1 \otimes A_2$ ويعرف بالرمز $u,v \in H_1 \otimes w, A(v \otimes z) = \langle u, A_1 v \rangle \langle w, A_2 z \rangle$

$$w, z, \in H_2$$

في هذه الرسالة ندرس بعض الخواص للجديدة التناسوري للمؤثرات المعروفة

على الفضاء $H_1,H_2 \otimes_H$ حيث $\otimes$ نشاط من $H_1 \otimes H_2$ نقوم بدراسة على العلاقات بين خواص المؤثرات $A_1 \otimes A_2$

التناسوري

لقد حللنا على عدة من النتائج الجديدة حسب علمنا وخاصة تلك التي تنتمي

الفصل الثاني
Introduction

Let $H$ be an infinite dimensional separable complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and let $B(H)$ be the algebra of all bounded linear operators on $H$, given $A_1, A_2 \in B(H)$, the tensor product $A_1 \otimes A_2$ on the space $H \hat{\otimes} H$ has been studied by many authors [10], [19], [33] and others.

The operation of taking tensor product $A_1 \otimes A_2$ preserves many properties of $A_1$ and $A_2 \in B(H)$ but by no means all of them. Thus, wherever $A_1 \otimes A_2$ is normal if and only if $A_1$ and $A_2 \in B(H)$ are. [19] and is similarly for hyponormal, subnormal, normaloid, $\theta$-operator and $U$-operator [10], [11], [19] it was shown in [34] that paranormal is not invariant under tensor product.

The operator $A$ is said to be strongly stable if $\|A^n x\| \to 0$ as $n \to \infty$ for all $x \in H$ [10].

If $A_1 \otimes A_2$ is strongly stable (and so necessarily power bounded) then at least one of $A_1$ and $A_2$ is strongly stable.

A generalized derivations operator $\delta_{A_1, A_2} : B(H) \to B(H)$ is a map defined by $\delta_{A_1, A_2}(X) = A_1 X - X A_2$ where $A_1$ and $A_2$ are operators on $B(H)$ and an elementary operator $\tau_{A_1, A_2} : B(H) \to B(H)$ is a map defined by $\tau_{A_1, A_2}(X) = A_1 X A_2$ where $A_1$ and $A_2$ are operators on $B(H)$.

In [15] Halmos, P.R. and Sunder, V.S. proved that there is an identification between $\tau_{A_1, A_2}(X)$ and $A_1 \otimes A_2^\ast$ and tried to study the property of $A_1 \otimes A_2^\ast$ by studying the property of $\tau_{A_1, A_2}(X)$.

The aim of this thesis is to study some properties of the operation of tensor product, study the connection between the operator $A_1 \otimes A_2 \in H \hat{\otimes} H$ and $A_1$ and $A_2$ and try to extend the class of operators that are invariant under this operation.
This thesis consists of three chapters.

In chapter one we recall general preliminaries in order to make the thesis as self contained as possible.

We recall the definition of tensor product of operators which is the main definition of our thesis. We study some of the basic properties that are needed later.

In chapter two we recall the definition of some kinds of operators and we give some of their properties which are needed later in this chapter then we study the properties of operators that are invariant under the operation of tensor product and the details of the proofs of several results which are stated without proofs.

At last we prove that $A_1 \otimes A_2$ is posinormal, $(2-2-8) \theta - adjo \int (2-2-11)$, binormal $(2-2-9)$ and pseudo operator$(2-2-10)$ if and only if each $A_i, i=1,2$ has thus property there result seem to be new to the best of our knowledge.

In chapter three we recall the definition of a compact operator and we study the compactness of tensor product of operators.

Also we discuss the identification between $\tau_{A_1,A_2}(X)$ and $A_1 \otimes A_2^*$ study the property of $A_1 \otimes A_2^*$ by studying the property of $\tau_{A_1,A_2}(X)$

In the last of this chapter we recall the definition of strongly stabile operator and uniformly stable operator and we study the relation between them and the strong stability of the operator $A_1 \otimes A_2$.

Again we give the proofs and the details of the proofs of several results which are stated without proofs and ones with hints of proofs.
Let $H$ be a complex Hilbert space, and let $B(H)$ denote the algebra of all bounded linear operators on $H$.

The purpose of this chapter is to explain certain terminology that is used throughout the thesis and to list some properties which are important for the discussion of our result.

This chapter consists of three sections.

In § 1-1 we deal with general preliminaries that we need later.

In § 1-2 we defined the tensor product of vector spaces in infinite dimensional space.

In § 1-3 we give the main definitions of our thesis and study some of their basic properties.
Let $H$ be a separable complex Hilbert space, and let $B(H)$ denote the algebra of all bounded linear operators on $H$.

Our goal in this chapter is to give basic properties of some kinds of operators and the relation of these operators with tensor product operators.

This chapter consists of two sections

In § 2-1 we recall some definitions, basic concepts, and some properties, which are important in the subsequent work.

In § 2-2 we study the relation between some kinds of operators with tensor product.
Chapter two

2.1 Some classes of operators

In this section we study some kind of operators with some relation between them which are needed in the next section; we start by giving the definition of subnormal operator.

Definition[2-1-1] [33, p539]

An operator $A$ is subnormal, if $A$ has a normal extension (i.e. there exists a Hilbert space $K$ containing $H$ as a subspace and a normal operator $B$ on $K$ such that $Ax = Bx$ for all $x \in H$).

The following proposition gives a relation between subnormal operators with hyponormal operators.

Proposition[2-1-2][30]

Every subnormal operator is hyponormal.

Proof:-

Let $A_1$ be a subnormal operator on $H$ and $A_2$ a normal extension of $A_1$ on $K$ then for $x, y \in H$

$\langle A_1^* x, y \rangle = \langle x, A_1 y \rangle = \langle x, A_2 Py \rangle = \langle PA_2^* x, y \rangle$

Where $P$ is the projection on $H$, then $A_1^* x = PA_2^* x \quad \forall x \in H$

So that

$\|A_1 x\| = \|A_2 x\| = \|A_2^* x\| \geq \|PA_2^* x\| = \|A_1^* x\| \quad \forall x \in H$

Definition[2-1-3] [10]

An operator $A$ on a Hilbert space $H$ is said to be quasinormal if $AA^* A = A^* A^2$.

As an illustration, we give the following example
Example:- [2-1-4]

Let $A$ be the unilateral shift operator defined on the Hilbert space $l_2(\mathbb{C})$ by

$$A(x_0, x_1, x_2, K) = (0, x_0, x_1, K)$$

Recall that $A^*(x_0, x_1, x_2, K) = (x_1, x_2, K)$

One can easily check that

$$AA^*(x_0, x_1, x_2, K) = A^*(0, x_0, x_1, K) = A(x_0, x_1, x_2, K) = (0, x_0, x_1, K)$$

$$A^*A^2(x_0, x_1, x_2, K) = A^*A(0, x_0, x_1, x_2, K) = A^*(0, 0, x_0, x_1, K) = (0, x_0, x_1, K)$$

hence $A$ is quasinormal operator.

Definition [2-1-5] [8]

An operator $A$ on a Hilbert space $H$ is said to be $K$–quasihyponormal if $A^*A^kA^k \geq 0$

Definition [2-1-6] [4]

An operator $A$ on a Hilbert space $H$ is said to be binormal operator if $A^*A$ commutes with $AA^*$. I.e. $[A^*A, AA^*] = 0$ we denote the class of binormal operators by $(BN)$.

Example:- [2-1-7]

Let $A$ be the unilateral shift operator defined on the Hilbert space $l_2(\mathbb{C})$ by

$$A(x_0, x_1, x_2, K) = (0, x_0, x_1, K)$$

recall that $A^*(x_0, x_1, x_2, K) = (x_1, x_2, K)$

One can easily check that

$$A^*A(x_0, x_1, x_2, K) = (x_0, x_1, x_2, K)$$

and

$$AA^*(x_0, x_1, x_2, K) = (0, x_1, x_2, K)$$

therefore $$(A^*A)(AA^*)x = (AA^*)(A^*A)x$$

for all $x = (x_0, x_1, x_2, K) \in l_2(\mathbb{C})$

Hence $A$ is a binormal operator.

The following proposition shows a relation between normal and quasinormal operators with binormal operators.
**Proposition [2-1-8] [13]**

1- Every normal operator is binormal operator  
2- Every quasinormal operator is binormal

**Proof:-**

1-Let $A$ be a normal operator, then  

Hence $A$ is a binormal operator.

2-Let $A$ be quasinormal operator, then  
\[A(A^*A) = (A^*A)A\]  
Implies that \((A^*A)A^* = A^*(A^*A)\). Now,  

Hence $A$ is a binormal operator.

**Definition [2-1-9] [5]**

An operator $A$ on a Hilbert space $H$ is called $\theta$–operator if $A^*A$ commutes with $A + A^*$. i.e. $[A^*A, A + A^*] = 0$

**Example:-[2-1-10]**

Let $A$ be the unilateral shift operator defined on the Hilbert space $l_2(\mathbb{C})$ by $A(x_0, x_1, x_2, \ldots) = (0, x_0, x_1, \ldots)$  
Can easily check that $((A^*A)(A + A^*)) = (x_1, x_0 + x_2, \ldots)$ and  
\[(A + A^*)(A^*A)(x_0, x_1, x_2, \ldots) = (x_1, x_0 + x_2, \ldots)\]

Which implies that $A$ is $\theta$–operator.

**Remark [2-1-11] [13]**

1- Every normal operator is a $\theta$–operator  
2- Every quasinormal operator is a $\theta$–operator
Proof:-

1-Let $A$ be a normal operator, then
Thus $A$ is a $\theta$-operator

2-Let $A$ be quasinormal operator, then
Hence $A$ is $\theta$-operator

**Definition** [2-1-12] [27]

Let $A \in B(H)$. $A$ is called a pseudo normal operator if $Ax = \lambda x$ for some $x \in H$, $\lambda \in \mathbb{C}$, then $A^* x = \overline{\lambda} x$, i.e. if $x$ is an eigen vector for $A$ with eigenvalue $\lambda$ then $x$ is an eigen vector for $A^*$ with eigenvalue $\overline{\lambda}$.

It is clear that if $\sigma_p(A) = \emptyset$ then $A$ is a pseudo normal operator.

**Example:**-[2-1-13]

The unilateral shift operator $A$ has no eigenvalue then $A$ is pseudo normal operator.

**Definition** [2-1-14] [21]

Let $A \in B(H)$. $A$ is called a $\ast$-paranormal operator, if
\[ \|A^* x\|^2 \leq \|A^2 x\| \text{ for every unit vector } x \in H. \]

**Example:**-[2-1-15]

Let $A$ be the unilateral shift operator defined on the Hilbert space $l_2(\mathbb{C})$ by
\[ A(y_1, y_2, K) = (0, y_1, y_2, K). \]
Recall that $A^* (y_1, y_2, K) = (y_2, y_3, K).$ Let $y = (y_1, 0, 0, K)$ one can easily check that
Chapter two

some properties of operators that are invariant under tensor product part I

\[ A^*y = A^*(y_1,0,K) = |0|^2 + |0|^2 + K = 0 \quad \text{And} \]

\[ A^2y = A(Ay) = A(0,y_1,o,K) = \sqrt{|0|^2 + |y_1|^2 + |0|^2 K} = |y_1|^2 = |y| \]

Suppose \( y \) is unit vector, i.e. \( |y| = 1 \)

Thus \( A \) is \(-paranormal\) operator.

**Remark** [2-1-16] [21]

Every \(-paranormal\) operator is pseudo normal operator.

**Proof:-**

Let \( Ax = \lambda x \), we may assume \( \|x\| = 1 \), since \( A \) is \(-paranormal\) operator, then

\[ A^*x \leq A^2x = A(\lambda x) = |\lambda|\|Ax\| = |\lambda|^2 \]

Thus \( A^*x \leq |\lambda|^2 \)

Now:-

\[ \left\| (A^* - \lambda I)x \right\|^2 = \left( (A^* - \lambda I)x, (A^* - \lambda I)x \right) \]

\[ = \left( A^*x, A^*x \right) - \left( \lambda x, A^*x \right) - \left( A^*x, \lambda x \right) + \left( \lambda x, \lambda x \right) \]

\[ = \left\| A^*x \right\|^2 - \lambda \left( \lambda x, Ax \right) + |\lambda|^2 \]

\[ \leq |\lambda|^2 - \lambda \left( \lambda x, Ax \right) + |\lambda|^2 \]

\[ \leq |\lambda|^2 - |\lambda|^2 + |\lambda|^2 = 0 \]

Therefore, \( \left\| (A^* - \lambda I)x \right\|^2 = 0 \) which implies, \( A^*x = \lambda x \).

**Definition** [2-1-17] [24]

If, \( A \in B(H) \), then \( A \) is posinormal operator if there exists an operator \( P \in B(H) \) such that \( AA^* = A^*PA \). \( P \) is called an interrupter of \( A \). The set of posinormal operator on \( H \) is denoted by \( P(H) \). \( A \) is called coposinormal if \( A^* \) is posinormal operator.
Chapter two
some properties of operators that are invariant under tensor product \[\text{part I}\]

Remark [2-1-18]

Every normal operator is posinormal operator, with \(P = 1\).

Example: [-2-1-19]

Consider the Hilbert space \(l_2(\mathcal{F})\)

Let \(C\) be Cesaro operator

\[
C = \begin{pmatrix}
1 & 0 & 0 & L \\
\frac{1}{2} & 1 & 0 & L \\
\frac{1}{3} & \frac{1}{3} & 1 & L \\
M & M & M & MO
\end{pmatrix}
\]

Regarded as an operator on \(H = l_2(\mathcal{F})\). The standard orthonormal basis for \(l_2(\mathcal{F})\) will be denoted by \(\{e_n : n = 0,1,2,K\}\)

If \(D\) is the diagonal operator with diagonal \(\left\{\frac{n+1}{n+2} : n = 0,1,2,L\right\}\), then \(D\) is:

\[
D = \begin{pmatrix}
\frac{1}{2} & 0 & 0 & L \\
0 & \frac{1}{2} & 0 & L \\
0 & 0 & \frac{3}{4} & L \\
M & M & M & MO
\end{pmatrix}
\]

Then \(C^*DC\) is

\[
C^*DC = \begin{pmatrix}
1 & 1 & \frac{1}{2} & L \\
\frac{1}{2} & 1 & \frac{1}{3} & L \\
\frac{1}{3} & \frac{1}{2} & 1 & L \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & L \\
M & M & M & MO
\end{pmatrix} = CC^*
\]

Thus the Cesaro operator is posinormal with interrupter \(D\).

Definition [2-1-20] [10]

Let \(A \in B(H)\), \(A\) is said to be \(P\)-hyponormal operator if \(\left(A^*A\right)^P \geq \left(AA^*\right)^P\) we denoted the class of \(P\)-hyponormal operators by \(H(P)\)
Remark [2-1-21]

It is clear that every hyponormal operator is \( P - \text{hyponormal} \).

Definition [2-1-22] [11]

The operator \( A \in B(H) \) is said to be belong to class \( U \) if \( A \) satisfies an absolute value condition \( |A^2| \geq |A|^2 \) where \( |A| = \left( A^* A \right)^{\frac{1}{2}} \) we denote the “class \( U \)” by simply \( U \)

Definition [2-1-23] [17]

An operator \( A \in B(H) \) is \( \theta - \text{adjoint} \) if \( A^* = e^{i\theta} A \) where \( \theta \in \mathbb{R} \)

Remarks [2-1-24]

1- If \( \theta = 0 \) then clearly \( A \) is \( 0 - \text{adjoint} \) if and only if \( A \) is Hermitian.
2- Let \( A \) be an \( \theta - \text{adjoint} \) operator, it is easy to verify that \( A \) is normal operator.

2.1. Tensor product of some classes of operators

In this section we study the operation of taking tensor products \( A_1 \otimes A_2 \otimes \ldots \otimes A_n \) preserver many properties of \( A_i \in B(H_i) \), but by no means all of them, thus the properties of normal, normal, posinormal, binormal, \( P - \text{hyponormal} \) and pseudo normal operator are invariant under tensor products. The \( * - \text{paranormal} \) is not invariant under tensor product.

The following theorem appeared in [19], it shows that if \( A_1, A_2, \ldots, A_n \) are linearly independent, then \( A_1 \otimes B_1 + K + A_n \otimes B_n = 0 \) if and only if \( B_1 = K = B_n = 0 \).
Let $A_1, A_2, K, A_n \in B(H_1)$ and $B_1, K, B_n \in B(H_2)$

**Theorem [2-2-1]**

If $A_1, A_2, K, A_n$ are linearly independent, then $A_1 \otimes B_1 + K + A_n \otimes B_n = 0$ if and only if $B_1 = K = B_n = 0$

**Proof:-**

Since $A_1, A_2, K, A_n$ are linearly independent, then $A_i$ is not linear combination of $A_2, K, A_n$ thus one can find vectors $x_1, x_2, K, x_r$ and $y_1, y_2, K, y_r$ in $H_1$, with $r(\infty)$ such that

$$\sum_{k=1}^{r} \langle A_i x_k, y_k \rangle = \begin{cases} 1 & i = 1; \\ 0 & i \neq 1. \end{cases}$$

Then for arbitrary vectors $y$ and $z$ in $H_2$ we have

$$\langle B_i y, z \rangle = \sum_{j=1}^{n} \sum_{k=1}^{r} \langle A_i x_k, y_k \rangle \langle B_i y, z \rangle$$

$$= \sum_{k=1}^{r} \langle \sum_{j=1}^{n} A_i \otimes B_i (x_k \otimes y), (y_k \otimes z) \rangle$$

Hence $\langle B_i y, z \rangle = 0$  $B_1 = 0$

Similarly we can prove that, $B_2 = K B_n = 0$ thus completing the proof.

In general If $A_1, A_2, K, A_n$ are linearly independent, then $A_1 \otimes B_1 + K + A_n \otimes B_n \leq 0$ if and only if $B_1 = K = B_n = 0$ [11]

The following theorem appeared in [19]

**Theorem [2-2-2]**

Let $A_i \in B(H_i), i = 1, 2, K, n$ and $A_1 \otimes A_2 \otimes L \otimes A_n \neq 0$ on the Hilbert space $H_1 \otimes H_2 \otimes L \otimes H_n$, $A_1 \otimes A_2 \otimes L \otimes A_n$ is a normal operator if and only if $A_i i = 1, 2, L, n$, are all normal operators.
Proof:-

By induction, it suffices to show that $A_1 \otimes A_2 \neq 0$ is normal if and only if both $A_1$ and $A_2$ are

\[ \Rightarrow \] Since $A_1 \otimes A_2$ is normal then

\[ (A_1 \otimes A_2)(A_1 \otimes A_2)^* = (A_1 \otimes A_2)^*(A_1 \otimes A_2) \]

i.e. $A_1 A_1^* \otimes A_2 A_2^* - A_1^* A_1 \otimes A_2^* A_2 = 0$

If $A_1 A_1^*$ and $A_1^* A_1$ are linearly independent then $A_2 A_2^* = A_2^* A_2 = 0$ hence $A_2 = 0$ contradiction hence $A_1 A_1^*$ and $A_1^* A_1$ are linearly dependent then $A_1 A_1^* = r A_1^* A_1$

Now

If $A_2 A_2^*$ and $A_2^* A_2$ are linearly independent then $A_1 A_1^* = A_1^* A_1 = 0$ hence $A_1 = 0$ contradiction then $A_2 A_2^*$ and $A_2^* A_2$ are linearly dependent and $A_2 A_2^* = r^{-1} A_2^* A_2$

\[ \|A_1\|^2 = \sup_{\|x\|=1} \langle x, A_1 A_1^* x \rangle \]

\[ = \sup_{\|x\|=1} \langle x, r A_1^* A_1 x \rangle \]

\[ = \sup_{\|x\|=1} r \langle x, A_1^* A_1 x \rangle = r \|A_1\|^2 \]

since $\|A_1\| = \|A_1^*\|$ then $r = 1$

and if $A_2 A_2^* = r^{-1} A_2^* A_2$

\[ \|A_2^*\|^2 = \sup_{\|x\|=1} \langle x, A_2 A_2^* x \rangle \]

\[ = \sup_{\|x\|=1} \langle x, r^{-1} A_2^* A_2 x \rangle \]

\[ = \sup_{\|x\|=1} r^{-1} \langle x, A_2^* A_2 x \rangle = r^{-1} \|A_2\|^2 \]

Since $\|A_1\| = \|A_1^*\|$ then $r^{-1} = 1$

Hence $A_1$ and $A_2$ are normal operator

$\Leftarrow$ On the other hand it is easy to show that if $A_1$ and $A_2$ are normal operators then $A_1 \otimes A_2$ is normal
Chapter two

some properties of operators that are invariant under tensor product part I

The following theorem appeared in [34]

**Theorem (2-2-3)**

Let \( A_1 \) and \( A_2 \) be non zero operators, then the tensor product \( A_1 \otimes A_2 \) is normaloid if and only if \( A_1 \) and \( A_2 \) are normaloid.

**Proof:-**

Suppose \( A_1 \) and \( A_2 \) are normaloid, then

\[
\| A_1 \otimes A_2 \| = \| A_1 \otimes A_2 \| = r(A_1 \otimes A_2) \\
\geq \sup \{ \xi : \xi \in \sigma(A_1 \otimes A_2) \} \\
= \sup \{ \| \lambda \| : \lambda \in \sigma(A_1), \mu \in \sigma(A_2) \} \\
= \| A_1 \| \| A_2 \|
\]

by theorem (1-3-9) thus \( A_1 \otimes A_2 \) is normaloid.

**Conversely**

Suppose that \( A_1 \otimes A_2 \) is normaloid. Then, again by theorem (1-3-9)

\[
\| A_1 \otimes A_2 \| = \sup \{ \xi : \xi \in \sigma(A_1 \otimes A_2) \} \\
= \sup \{ \| \lambda \| : \lambda \in \sigma(A_1), \mu \in \sigma(A_2) \} \\
\leq r(A_1)r(A_2) \\
\leq \| A_1 \| \| A_2 \|
\]

Since \( r(A_1) \leq \| A_1 \| \) and \( r(A_2) \leq \| A_2 \| \), we have \( r(A_1) = \| A_1 \| \) and \( r(A_2) = \| A_2 \| \).

In general one can prove that \( A_1 \otimes A_2 \otimes \cdots \otimes A_n \neq 0 \) an operator on a Hilbert space \( H_1 \otimes H_2 \otimes \cdots \otimes H_n \), is normaloid operator if and only if \( A_i, i = 1, 2, \ldots, n \), are all normaliod operators.
Chapter two

some properties of operators that are invariant under tensor product part 1

The following theorem appeared in [19], we give the details of the proof.

**Theorem-[2-2-4]**

Let \( A_i \in B(H_1), i = 1, 2, K, n \) and \( A_1 \otimes A_2 \otimes L \otimes A_n \neq 0 \) on the Hilbert space \( H_1 \otimes H_2 \otimes L \otimes H_n \), \( A_1 \otimes A_2 \otimes L \otimes A_n \) is a hyponormal operator if and only if \( A_i \ i = 1, 2, L, n \), are all hyponormal operators.

**Proof:-**

By induction, it suffices to show that \( A_1 \otimes A_2 \neq 0 \) is hyponormal if and only if both \( A_1 \) and \( A_2 \) are.

\( \Rightarrow \) Let \([A_1] = A_1 A_1^* - A_1^* A_1 \) and \([A_2] = A_2 A_2^* - A_2^* A_2 \)

Since \((A_1 \otimes A_2)\) is hyponormal operator then

\[
(A_1 \otimes A_2)(A_1 \otimes A_2)^* - (A_1 \otimes A_2)^*(A_1 \otimes A_2) = A_1 A_1^* \otimes A_2 A_2^* - A_1^* A_1 \otimes A_2 A_2^* \\
= A_1 A_1^* \otimes A_2 A_2^* - A_1^* A_1 \otimes A_2 A_2^* + A_1^* A_1 \otimes A_2 A_2^* - A_1^* A_1 \otimes A_2 A_2^* \\
= (A_1 A_1^* - A_1^* A_1) \otimes A_2 A_2^* + A_1^* A_1 \otimes (A_2 A_2^* - A_2^* A_2) \\
= [A_1] \otimes A_2 A_2^* + A_1^* A_1 \otimes [A_2] \geq 0.
\]

\( \cdots (2-1) \)

For any \( x \in H_1 \) and \( y \in H_2 \)

\[
\langle (A_1 A_1^* - A_1^* A_1) \otimes A_2 A_2^* + A_1^* A_1 \otimes (A_2 A_2^* - A_2^* A_2) \rangle (x \otimes y), (x \otimes y) \rangle = \\
\langle (A_1 A_1^* - A_1^* A_1) \otimes A_2 A_2^* \rangle (x \otimes y), (x \otimes y) \rangle + \langle (A_1^* A_1 \otimes (A_2 A_2^* - A_2^* A_2) \rangle (x \otimes y), (x \otimes y) \rangle = \\
\langle (A_1 A_1^* - A_1^* A_1) x, x \rangle \langle (A_2 A_2^* y, y) \rangle + \langle (A_1^* A_1 x, x) \rangle \langle (A_2 A_2^* - A_2^* A_2) y, y \rangle = \\
\langle [A_1] x, x \rangle \|A_2^* y\|^2 + \|A_2^* y\|^2 \langle [A_2] y, y \rangle \geq 0.
\]

\( \cdots (2-2) \)

If \( A_2 \) is not hyponormal there must be a vector \( y_0 \in H_2 \) such that \( \langle [A_2] y_0, y_0 \rangle = -s \langle 0, 0 \rangle \) obviously \( \|A_2^* y_0\|^2 = r \neq 0 \) since

\[
\langle (A_1 A_1^* - A_1^* A_1) x, x \rangle = \langle A_1 A_1^* x, x \rangle - \langle A_1^* A_1 x, x \rangle \\
= \langle A_1 A_1^* x, x \rangle - \langle A_1^* A_1 x, x \rangle \\
= \|A_1^* x\|^2 - \|A_1 x\|^2
\]

35
Chapter two

Some properties of operators that are invariant under tensor product

Part I

So it follows from (2-2)

\[-s\|A_1 x\|^2 + r\left(\|A_1^* x\|^2 - \|A_1 x\|^2\right) \geq 0\]

\[-s\|A_1\|^2 + r\|A_1^*\|^2 - \|A_1\|^2 \geq 0\]

This implies \((r - s)\|A_1\|^2 \geq r\|A_1\|^2\) being impossible. Hence \(A_i\) must be hyponormal.

Similarly by

If \([A_1] = A_1^* A_1 - A_1 A_1^*\) and \([A_2] = A_2^* A_2 - A_2 A_2^*\)

Then \([A_1] \otimes A_2^* A_2 + A_1 A_1^* \otimes [A_2] \geq 0\) \(\ldots (2-3)\)

By (2-3) for any \(x \in H_1\) and \(y \in H_2\).

\[\langle [A_1] x, x \rangle \|A_2 y\|^2 + \|A_1^* x\|^2 \langle [A_2] y, y \rangle \geq 0\] \(\ldots (2-4)\)

By the same way we can prove \(A_i\) is hyponormal operator, too

\(\Leftarrow\) On the other hand see theorem (1-3-12)

**Remark** [2-2-5]

\[|A_1 \otimes A_2|^{2p} = |A_1|^{2p} \otimes |A_2|^{2p}\] Where \(p = 1, 2, 3, K\)

**Proof:** - see [11]

The following theorem appeared in [10]

**Theorem** [2-2-6]

\(A_1 \otimes A_2 \in H(P) \iff A_1 \text{ and } A_2 \in H(P)\)

**Proof:** -

If \(A_1\) and \(A_2 \in H(P)\) according to definition (2-1-20)
some properties of operators that are invariant under tensor product part I

\[(A^*_1 A_1)^p \geq (A_1 A^*_1)^p\] \(\text{And} (A^*_2 A_2)^p \geq (A_2 A^*_2)^p\)

\[
\left( (A_1 \otimes A_2)^*(A_1 \otimes A_2) \right)^p - \left( (A_1 \otimes A_2)(A_1 \otimes A_2)^* \right)^p
\]

\[
= (A^*_1 A_1 \otimes A^*_2 A_2)^p - (A^*_1 A_1 \otimes A^*_2 A_2)^p
\]

\[
= (A^*_1 A_1)^p \otimes (A^*_2 A_2)^p - (A^*_1 A_1)^p \otimes (A^*_2 A_2)^p + (A^*_1 A_1)^p \otimes (A^*_2 A_2)^p - (A^*_1 A_1)^p \otimes (A^*_2 A_2)^p
\]

\[
= \left( (A_1 A_1)^p - (A_1 A_1)^p \right) \otimes (A^*_2 A_2)^p + (A^*_1 A_1)^p \otimes \left( (A^*_2 A_2)^p - (A^*_2 A_2)^p \right) \geq 0
\]

Hence \(A_1 \otimes A_2\) is a hyponormal operator.

\(\Leftrightarrow\) Uses spectral decomposition, which are is out said the scope of our work, so that we are not going to go through.

The following theorem appeared in [11]

**Theorem [2-2-7]**

For non-zero \(A_1, A_2 \in B(H), A_1 \otimes A_2 \in U\) if and only if \(A_1\) and \(A_2 \in U\)

**Proof:-**

Suppose that \(A_1 \otimes A_2 \in U\) then according to definition (2-1-22)

\[
|A_1|^2 \otimes |A_2|^2 = |A_1 \otimes A_2|^2
\]

\[
\leq \left| (A_1 \otimes A_2)^2 \right|
\]

\[
= |A_1^2 \otimes A_2^2|
\]

\[
= |A_1^2| \otimes |A_2^2|
\]

Hence, by theorem (2-2-1) there exists a scalar \(c \rangle 0\) such that \(\|A_1\|^2 \leq c |A_1|^2\) and \(\|A_2\|^2 \leq c^{-1} |A_2|^2\) this implies that

\[
\|A_1\|^2 = \sup_{\|x\|=1} \langle |A_1|^2 x, x \rangle
\]

\[
\leq \sup_{\|x\|=1} \langle c |A_1|^2 x, x \rangle
\]

\[
\leq c \|A_1^2\|
\]

\[
= c \|A_1\| ^2
\]

\[
\leq c \|A_1\|^2
\]
Chapter two

some properties of operators that are invariant under tensor product part I

\[\|A_2\|^2 = \sup_{\|x\|=1}\langle |A_2|^2 x, x \rangle\]
\[\leq \sup_{\|x\|=1}\langle c^{-1}|A_2|^2 x, x \rangle\]
\[\leq c^{-1}\|A_2\|^2\]
\[= c^{-1}\|A_2\|^2\]
\[\leq c^{-1}\|A_2\|^2\]

Clearly, we must have \( c = 1 \), and then \( A_1, A_2 \in U \)

Conversely, if \( A_1, A_2 \in U \), then
\[
\left(\left(\|A_1^2\| \otimes |A_2|^2\right) - |A_1|^2 \otimes |A_2|^2\right)
\geq |A_1|^2 \otimes |A_2|^2 + |A_1|^2 \otimes |A_2|^2 - 2|A_1|^2 \otimes |A_2|^2
\]
\[
= \left(|A_1|^2 - |A_1|^2\right) \otimes |A_2|^2 + |A_1|^2 \otimes \left(|A_2|^2 - |A_2|^2\right) \geq 0
\]

Hence \( A_1, A_2 \in U \)

In general one can prove the following theorem

**Theorem [2-2-8]**

Let \( A_i \) s be non-zero operators, then \( A_1 \otimes A_2 \otimes L \otimes A_n \) an operator on the Hilbert space \( H_1 \otimes H_2 \otimes L \otimes H_n \) is \( U \) – operator if and only if each \( A_i \) is \( U \) – operator.

In the following theorems we prove that there exist other operator properties that are invariant under tensor product.

**Theorem [2-2-9]**

Let \( A_i \in B(H_i), i = 1,2,L,n \) and \( A_1 \otimes A_2 \otimes L \otimes A_n \neq 0 \) on the Hilbert space \( H_1 \otimes H_2 \otimes L \otimes H_n \), \( A_1 \otimes A_2 \otimes L \otimes A_n \) is a posinormal operator if and only if \( A_i \) \( i = 1,2,L,n \), are all posinormal operators.
Proof:-

By induction, it suffices to show that if $A_1 \otimes A_2 \neq 0$ then $A_1 \otimes A_2$ is posinormal if and only if both $A_1$ and $A_2$ are. Let $A_1 \otimes A_2 \neq 0$ is posinormal operator, according to definition (2-1-17) then

$$(A_1 \otimes A_2)(A_1 \otimes A_2)^* = (A_1 \otimes A_2)^*(P_1 \otimes P_2)(A_1 \otimes A_2)$$

$A_1A_1^* \otimes A_2A_2^* - A_1^* P_1A_1 \otimes A_2^* P_2A_2 = 0$

If $A_1A_1^*$ and $A_1^* P_1A_1$ are linearly independent then $A_2A_2^* = A_2^* P_2A_2 = 0$ then $A_2 = 0$ contradiction

Hence $A_1A_1^*$ and $A_1^* P_1A_1$ are linearly dependent then $A_1A_1^* = rA_1^* P_1A_1$.

If $A_2A_2^*$ and $A_2^* P_2A_2$ are linearly independent $A_1A_1^* = A_1^* P_1A_1 = 0$ then $A_1 = 0$ contradiction

Hence $A_2A_2^*$ and $A_2^* P_2A_2$ are linearly dependent then $A_2A_2^* = r^{-1}A_2^* P_2A_2$.

Now to prove $r = 1$

$$\|A_1\|^2 = \|A_1A_1^*\| = \|A_1^* P_1A_1\| \leq |r| \|A_1^*\| A_1 = |r| \|A_1\| = |r| \|A_1\|^2.$$ hence $1 \leq |r|$ and

$$\|A_2\|^2 = \|A_2A_2^*\| = \|A_2^* P_2A_2\| \leq |r^{-1}| \|A_2^*\| A_2 \| = |r^{-1}| \|A_2\| = |r^{-1}| \|A_2\|^2.$$ hence $1 \leq |r^{-1}|$ it implies that $r = 1$

$\iff$ On the other hand it is easy to see that if $A_1$ and $A_2$ are posinormal operators then $A_1 \otimes A_2$ is posinormal.

**Theorem** (2-2-10)

Let $A_i \in B(H_i), i=1,2,\ldots,n$ and $A_1 \otimes A_2 \otimes L \otimes A_n$ on Hilbert space $H_1 \otimes H_2 \otimes L \otimes H_n$, $A_1 \otimes A_2 \otimes L \otimes A_n$ is binormal operator if and only if each $A_i$, $i=1,2,\ldots,n$, is binormal operator.
Proof:-

By induction, it suffices to show that if $A_1 \otimes A_2$ is binormal if and only if both $A_1$ and $A_2$ are.

$\Leftarrow$ Suppose that $A_1 \otimes A_2 \neq 0$ is binormal operator, according to definition (2-1-6) then

$$(A_1 \otimes A_2)^*(A_1 \otimes A_2)^2(A_1 \otimes A_2)^* = (A_1 \otimes A_2)(A_1 \otimes A_2)^2(A_1 \otimes A_2)$$

$$A_1^* A_1^* A_1^* \otimes A_2^* A_2^* A_2^* - A_1 A_1^* A_1 A_2 A_2^* A_2 = 0$$

If $A_1^* A_1^* A_1^*$ and $A_1 A_1^* A_1$ are linearly independent then $A_2^* A_2^* A_2^* A_2 = 0$ then $A_2 = 0$ contradiction. Hence $A_1^* A_1^* A_1^*$ and $A_1 A_1^* A_1$ are linearly dependent then $A_1^* A_1^* A_1^* = r A_1 A_1^* A_1$.

Similarly if $A_2^* A_2^* A_2^*$ and $A_2 A_2^* A_2$ are linearly independent then $A_1^* A_1^* A_1^* = A_1 A_1^* A_1 = 0$ hence $A_1 = 0$ contradiction. Hence $A_2^* A_2^* A_2^*$ and $A_2 A_2^* A_2$ are linearly dependent then $A_2^* A_2^* A_2^* A_2 = r^{-1} A_2 A_2^* A_2$

Now it must be proving that $r = 1$.

$$\| A_1^* A_1^* A_1^* \| = \| r (A_1 A_1^* A_1) \|$$

$$= \| r A_1 A_1^* A_1 \|$$

$$= \| r (A_1^* A_1^* A_1^*)^* \|$$

Hence $|r| = 1$

Similarly

$$\| A_2^* A_2^* A_2^* \| = \| r^{-1} (A_2 A_2^* A_2) \|$$

$$= \| r^{-1} A_2 A_2^* A_2 \|$$

$$= \| r^{-1} (A_2^* A_2^* A_2^*)^* \|$$

Hence $|r^{-1}| = 1$

implies that $r = 1$ hence $A_1$ and $A_2$ are binormal operators.

$\Rightarrow$ on the other hand it is easy to see that if $A_1$ and $A_2$ are binormal operators then $A_1 \otimes A_2$ is binormal operator.
Chapter two

some properties of operators that are invariant under tensor product part I

**Theorem (2-2-11)**

Let $A_i \in B(H_i), i = 1, 2, K, n$ and $A_1 \otimes A_2 \otimes L \otimes A_n \neq 0$ on $H_1 \otimes H_2 \otimes L \otimes H_n$, $A_1 \otimes A_2 \otimes L \otimes A_n \neq 0$ is pseudo normal operator if and only if each $A_i, i = 1, 2, L, n$, is pseudo normal operator.

**Proof:-**

By induction, it suffices to show that $A_1 \otimes A_2 \neq 0$ is pseudo normal if and only if both $A_1$ and $A_2$ are.

Let $A_1 \otimes A_2 \neq 0$ be pseudo normal operator, according to definition (2-1-12)

Let $A_1 x = \lambda x$ and $A_2 y = \mu y$

If $(A_1 \otimes A_2)(x \otimes y) = \lambda \mu (x \otimes y)$ … (2-5)

Then $(A_1 \otimes A_2)^* (x \otimes y) = \lambda \mu^* (x \otimes y)$ … (2-6)

In (2-5) if $A_1 x, \lambda x$ are linearly independent then $A_2 y = \mu y = 0$ contradiction hence $A_1 x, \lambda x$ are linearly dependent $A_1 x = t \lambda x$ but $A_1 x = \lambda x$ then $t = 1$

In (2-5) $A_2 y, \mu y$ are linearly independent then $A_1 x = \lambda x = 0$ contradiction

Hence $A_2 y, \mu y$ are linearly independent then $A_2 y = t^{-1} \mu y$ but $A_2 y = \mu y$ then $t^{-1} = 1$

In (2-6) if $A_1^* x$ and $\lambda x$ are linearly independent then $A_2^* y = \mu y = 0$ then $A_2^* = 0$ contradiction.

Hence $A_1^* x$ and $\lambda x$ are linearly dependent $A_1^* x = r \lambda x$

Similarly

If $A_2^* y$ and $\mu y$ are linearly independent then $A_1^* x = \lambda x = 0$ then $A_1^* = 0$ contradiction. Hence $A_2^* y$ and $\mu y$ are linearly dependent $A_2^* y = r^{-1} \mu y$

It is clear $t = r = 1$

$\Rightarrow$ It is clear that if $A_1$ and $A_2$ are pseudo normal operators then $A_1 \otimes A_2$ is pseudo normal operator.
Chapter two

some properties of operators that are invariant under tensor product part I

Theorem [2-2-12]

Let \( A_i \in B(H_i), i = 1,2,K,n \) and \( A_1 \otimes A_2 \otimes L \otimes A_n \neq 0 \) on 
\( H_1 \otimes H_2 \otimes L \otimes H_n \) if each \( A_i, i = 1,2,L,n \), is \( \theta – adjo \) int operator 
then \( A_1 \otimes A_2 \otimes L \otimes A_n \neq 0 \) is \( \theta – adjo \) int operator.

Proof:-

By induction, it suffices to show that \( A_1 \otimes A_2 \neq 0 \) is \( \theta – adjo \) int if \( A_1 \) 
and \( A_2 \) are \( \theta – adjo \) int operators.

If \( A_1^* = e^{i\theta} A_1 \) and \( A_2^* = e^{i\theta_2} A_2 \) it is easy to see that \( (A_1 \otimes A_2) \) is 
\( \theta_1 + \theta_2 – adjo \) int operator.

Proposition [2-2-13]

If \( A_1 \) and \( A_2 \) are \( *–paranormal \) operators then \( A_1 \otimes A_2 \) 
is \( *–paranormal \)

Proof

Since \( A_1 \) and \( A_2 \) are \( *–paranormal \) operators according to definition (2-1-14), then

\[
\|A_1^2 x\| \geq \|A_1 x\|^2 \quad \text{and} \quad \|A_2^2 y\| \geq \|A_2 y\|^2
\]

\[
\|A_1^2 x\| \|A_2^2 y\| \geq \|A_1^* x\|^2 \|A_2^* y\|^2
\]

\[
\|(A_1 \otimes A_2)^2 (x \otimes y)\| \geq \|(A_1 \otimes A_2)^* (x \otimes y)\|^2
\]

Hence \( A_1 \otimes A_2 \) is \( *–paranormal \)

Remark [2-2-14]

If \( A_1 \otimes A_2 \) is \( *–paranormal \) operator, then it may not be true that also 
\( A_1 \) and \( A_2 \) are \( *–paranormal \) operators, for example:-
Let $H = L^2(\mathbb{C})$, $A : H \to H$ define as follows:

$A_1(x_1, x_2, K) = (0, 0, x_2, 0, K)$

it is easily checked that

$A_1^*(x_1, x_2, K) = (0, x_3, 0, 0, K)$

Let $x = (x_1, 0, x_3, x_4, K)$, then $A_1x = (0, 0, 0, K)$

$A_1^*x = (0, x_3, 0, K)$  \[ \|A_1^*x\| = |x_3|^2 \]

It is clear that $A_1$ is not $\ast$-$\text{paranormal}$ operator; let $A_2$ be defined as operator $A$ in example (2.1-15)

\[
\|A_2 \otimes A_2^* (x \otimes y)\| = \|A_2^*x\| \|A_2^*y\| = 0 \cdot |x_1|^2 = 0
\]

\[
\|A_1^* \otimes A_2^* (x \otimes y)\| = \|A_1^*x\| \|A_2^*y\| = |x_3|^2 \cdot 0 = 0
\]

Hence $A_1 \otimes A_2$ is $\ast$-$\text{paranormal}$ operator

The following theorem appeared in [19], we give the details of the proof.

**Theorem [2-2-15]**

Let $A_i \in B(H), i = 1, 2, K, n$ and $A_1 \otimes A_2 \otimes L \otimes A_n \neq 0$ on $H_1 \hat{\otimes} H_2 \hat{\otimes} L \hat{\otimes} H_n$ then:

1- $A_1 \otimes A_2 \otimes L \otimes A_n$ is self–adjoint if and only if there exist real numbers $\alpha_1, \alpha_2, K, \alpha_n$ with $\alpha_1 + \alpha_2 + K + \alpha_n = 0$ such that $e^{i\alpha_k} A_k$ is self–adjoint for each $k = 1, 2, K, n$.

2- $A_1 \otimes A_2 \otimes L \otimes A_n$ is unitary (resp. (isometry) if and only if there exist positive numbers $\alpha_1, \alpha_2, K, \alpha_n$ with $\alpha_1 \cdot \alpha_2 \cdot K \cdot \alpha_n = 1$ such that $a_k A_k$ is unitary (resp. (isometry) for each $k = 1, 2, K, n$.

**Proof:-**

1- By induction, it suffices to show that $A_1 \otimes A_2 \neq 0$ is self-adjoint if and only if $e^{i\alpha_1} A_1$ and $e^{i\alpha_2} A_2$ are self-adjoint.

Suppose that $e^{i\alpha_1} A_1$ and $e^{i\alpha_2} A_2$ are self-adjoint,

$e^{i\alpha_1} A_1 = e^{-i\alpha_1} A_1^*$ and $e^{i\alpha_2} A_2 = e^{-i\alpha_2} A_2^*$.
Chapter two

Some properties of operators that are invariant under tensor product part I

\[ e^{i(a_1 + a_2)}(A_1 \otimes A_2) = e^{i \theta_1} A_1 \otimes e^{i \theta_2} A_2 \]
\[ = e^{-i \theta_1} A_1^\ast \otimes e^{-i \theta_2} A_2^\ast \]
\[ = e^{-i(a_1 + a_2)}(A_1^\ast \otimes A_2^\ast) \]

Since \( a_1 + a_2 = 0 \) then \((A_1 \otimes A_2) = (A_1 \otimes A_2)^\ast\)

Suppose that \( A_1 \otimes A_2 \) is self-adjoint operator

\[ e^{i \theta_1} A_1 \otimes e^{i \theta_2} A_2 = e^{i(a_1 + a_2)}(A_1 \otimes A_2) \]
\[ = e^{-i(a_1 + a_2)}(A_1^\ast \otimes A_2^\ast) \]
\[ = e^{-i \theta_1} A_1^\ast \otimes e^{-i \theta_2} A_2^\ast \]
\[ e^{i \theta_1} A_1 \otimes e^{i \theta_2} A_2 = e^{-i \theta_1} A_1^\ast + e^{-i \theta_2} A_2^\ast \]
\[ e^{i \theta_1} A_1 \otimes e^{i \theta_2} A_2 - e^{-i \theta_1} A_1^\ast \otimes e^{-i \theta_2} A_2^\ast = 0 \quad \text{By theorem (2-2-1)} \]

If \( e^{i \theta_1} A_1 \) and \( e^{-i \theta_1} A_1^\ast \) are linearly independent then \( e^{i \theta_2} A_2 = e^{-i \theta_2} A_2^\ast = 0 \)

hence \( A_2 = 0 \) contradiction then \( e^{i \theta_1} A_1 \) and \( e^{-i \theta_1} A_1^\ast \) are linearly dependent

\( e^{i \theta_1} A_1 = re^{-i \theta_1} A_1^\ast \)

Similarly

If \( e^{i \theta_2} A_2 \) and \( e^{-i \theta_2} A_2^\ast \) are linearly independent then \( e^{i \theta_1} A_1 = e^{-i \theta_1} A_1^\ast = 0 \)

hence \( A_1 = 0 \) contradiction then \( e^{i \theta_2} A_2 \) and \( e^{-i \theta_2} A_2^\ast \) are linearly dependent

\( e^{i \theta_2} A_2 = r^{-1} e^{-i \theta_2} A_2^\ast \)
\[ A_1 = e^{-2i \theta_2} A_1^\ast \quad r = e^{-2i \theta_1} \ln r = -2i \theta_1 \]
\[ A_2 = e^{-2i \theta_2} A_2^\ast \quad r^{-1} = e^{-2i \theta_2} \ln r^{-1} = -2i \theta_2 \]
\[ a_1 = \frac{i \ln r}{2} \quad a_2 = \frac{i \ln r^{-1}}{2} \quad r = \frac{1}{r} \quad \ln r = \ln r^{-1} \]
\[ a_1 + a_2 = \frac{i \ln r}{2} + \frac{i \ln r^{-1}}{2} = 0 \]

2- Suppose that \( a_1 A_1 \) and \( a_2 A_2 \) are unitary operators then
\[ a_1 A_1 a_1^\ast = a_1 A_1^\ast a_1 A_1 = I_1 \quad a_1^2 A_1 A_1^\ast = a_1^2 A_1^\ast A_1 = I_1 \quad \text{then} \]
\[ A_1 A_1^\ast = A_1^\ast A_1 = I_1 \]
\[ a_2 A_2 a_2^\ast = a_2 A_2^\ast a_2 A_2 = I_2 \quad a_2^2 A_2 A_2^\ast = a_2^2 A_2^\ast A_2 = I_2 \quad \text{then} \]
\[ A_2 A_2^\ast = A_2^\ast A_2 = I_2 \]
\[ A_1 A_1^\ast \otimes A_2 A_2^\ast = I_1 \otimes I_2 \quad \text{and} \quad A_1^\ast A_1 \otimes A_2^\ast A_2 = I_1 \otimes I_2 \]
\[ (A_1 \otimes A_2)(A_1 \otimes A_2)^\ast = I_1 \otimes I_2 \quad \text{and} \quad (A_1 \otimes A_2)^\ast (A_1 \otimes A_2) = I_1 \otimes I_2 \]

Hence \((A_1 \otimes A_2)\) is unitary operator.
Now suppose that \((A_1 \otimes A_2)\) is unitary operator then
\[
(A_1 \otimes A_2)(A_1 \otimes A_2)^* = I_1 \otimes I_2 \quad \text{and} \quad (A_1 \otimes A_2)^*(A_1 \otimes A_2) = I_1 \otimes I_2
\]
\[
(A_1 \otimes A_2)(A_1 \otimes A_2)^* - I_1 \otimes I_2 = 0 \quad A_1A_1^* \otimes A_2^*A_2 - I_1 \otimes I_2 = 0 \quad \ldots \quad (2-7)
\]
and
\[
(A_1 \otimes A_2)^*(A_1 \otimes A_2) - I_1 \otimes I_2 = 0 \quad A_1^*A_1 \otimes A_2^*A_2 - I_1 \otimes I_2 = 0 \quad \ldots \quad (2-8)
\]

In (2-7)
If \(A_1A_1^*, I_1\) are linearly independent then \(A_2A_2^* = I_2 = 0\) then \(A_2 = 0\) contradiction, hence \(A_1A_1^*, I_1\) are linearly dependent then \(A_1A_1^* = rI_1\).
And if \(A_2A_2^*, I_2\) are linearly independent then \(A_1A_1^* = I_1 = 0\) then \(A_1 = 0\) contradiction hence \(A_2A_2^*, I_2\) are linearly dependent then \(A_2A_2^* = r^{-1}I_2\).

Similarly
In (2-9) if \(A_1^*A_1, I_1\) are linearly independent then \(A_2^*A_2 = I_2 = 0\) then \(A_2 = 0\) contradiction, hence \(A_1^*A_1, I_1\) are linearly dependent then
\(A_1^*A_1 = tI_1\) and if \(A_2^*A_2, I_2\) are linearly independent then
\(A_1^*A_1 = I_1 = 0\) then \(A_1 = 0\) contradiction hence \(A_2^*A_2, I_2\) are linearly dependent then \(A_2^*A_2 = t^{-1}I_2\).

Since \(A_1^*A_1 = rt^{-1}A_1A_1^*\) let \(a_1 = r\) \(\frac{1}{2}\) and \(a_2 = a_1^{-1}\) then \(a_1a_2 = 1\) then both \(a_1A_1\) and \(a_2A_2\) are unitary.

The following theorem appeared in [19]

**Theorem [2-2-16]**
\(A_1 \otimes A_2 \otimes L \otimes A_n\) is \(k\)-quasihyponormal if and only if one of the following holds:
1. Every \(A_i\) is \(k\)-quasihyponormal for \(i = 1, 2, K, n\)
2. \(A_i^k = 0\) for some \(i\) \(1 \leq i \leq n\)

**Proof:-**
By induction, it suffices to show that \(A_1 \otimes A_2\) is \(k\)-quasihyponormal operator if and only if \(A_i\) and \(A_2\) are \(k\)-quasihyponormal operators.
Suppose that \((A_1 \otimes A_2)\) is \(k\)-quasihyponormal operator according to definition (2-1-5), then
\[
(A_1 \otimes A_2)^* (A_1 \otimes A_2)(A_1 \otimes A_2)^k \geq 0
\]
and
\[
(A_1 \otimes A_2)^* (A_1 \otimes A_2)(A_1 \otimes A_2)^* \leq 0
\]
Let \(x \in H_1\) and \(y \in H_2\), then
\[
\langle A_1^k [A_1] A_1^k \otimes A_2^{k+1} A_2 + A_1^k A_1^* A_1^k \otimes A_2^k [A_2] A_2^k \rangle (x \otimes y) (x \otimes y) \geq 0
\]... (2-9)
Suppose that \(A_1^k \neq 0\) and \(A_2^k \neq 0\), if \(A_i\) is not \(k\)-quasihyponormal, then there is a vector \(x_0 \in H_1\) for which
\[
\langle A_1^k [A] A^k x_0, x_0 \rangle = -s \langle 0
\]
and
\[
\|A_1^k A_1^k x_0\|^2 = r \neq 0 \text{ from (2-11) we get}
\]
\[
(r - s) \|A_2^{k+1} y\|^2 \geq r \|A_2^k A_2 y\|^2
\]... (2-12)
For all \(y \in H_2\). Let \(B_1 = A_2 \bigg|_{\text{ran} A_2^k}\) and define \(E \in B(\text{ran} A_2^k \rightarrow H_2)\) by
\[
Ez = A_2^k z
\]it is easily seen that \(\|E\| \leq \|B_1\|\). but (2-12) implies
some properties of operators that are invariant under tensor product. Part I

\[(r - s)\|B\|^s \geq r\|E\|^s\], impels a contradiction. Hence \(A_i\) is \(k\)-quasihyponormal.

A similar argument we can prove \(A_2\) is \(k\)-quasihyponormal where chose
\[\{A_1\} = A_1 A_i^* - A_i^* A_1\quad,\quad \{A_2\} = A_2 A_2^* - A_2^* A_2\]
and \[\{A_1 \otimes A_2\} = (A_1 \otimes A_2)(A_1 \otimes A_2)^* - (A_1 \otimes A_2)^*(A_1 \otimes A_2)\]
\[
\iff \quad \text{If } A_1, A_2 \text{ are } k\text{-quasihyponormal}
\]
Then \(A_i^k [A_1 A_i^k \geq 0 \text{ and } A_2^k [A_2^k A_i^k \geq 0
\]

\[
A_i^k \left( A_1 A_i^* - A_i^* A_1 \right) A_i^k = A_i^k A_1 A_i^* A_i^k - A_i^k A_i^k A_i^k \geq 0
\]
\[
A_2^k \left( A_2 A_2^* - A_2^* A_2 \right) A_2^k = A_2^k A_2 A_2^* A_2^k - A_2^k A_2^k A_2^k \geq 0
\]

Then it is easily seen that \((A_1 \otimes A_2)\) is \(k\)-quasihyponormal operator.

2- It is clear.

The following theorem gives the relation between \(\theta - \text{operator}\) of tensor product and normal operator.

The following theorem appeared in [19] without proof and we give a proof.

**Theorem** [2-2-17]

\(A_1 \otimes I_2 + I_1 \otimes A_2\) is a \(\theta - \text{operator}\) if and only if \(A_1\) and \(A_2\) are normal operators.

**Proof:**

Suppose that \(A_1 \otimes I_2 + I_1 \otimes A_2\) is \(\theta - \text{operator}\), according to definition (2-1-9) then
\[A_i A_i^* A_i \otimes I_2 + A_1 \otimes A_2^* A_2 + I_1 \otimes A_2 A_2^* A_2 + A_1 A_i^* A_i \otimes I_2 + A_1 A_i^* A_i \otimes A_2^* A_2^* + A_1^* \otimes A_2 A_2^* A_2 + I_1 \otimes A_2 A_2^* A_2 - A_1 \otimes A_2 A_2^* A_2 - I_1 \otimes A_2 A_2^* A_2 = 0
\]

if

\(A_i A_i^* A_i, A_i^* A_i, A_1, A_i^* A_i, A_i^* A_i, A_i^* A_i, A_i^* A_i, A_i^* A_i\) are linearly independent then
\(I_2 = A_2 A_2^* A_2 = A_2 A_2^* A_2 = A_2 A_2^* A_2 = A_2 A_2^* A_2 = A_2 A_2^* A_2 = A_2 A_2^* A_2 = 0
\)
Contradiction

Hence \(A_i A_i^* A_i, A_i^* A_i, A_i^* A_i, A_i^* A_i, A_i^* A_i, A_i^* A_i, A_i^* A_i, A_i^* A_i\) are linearly dependent then it is clear \(A_i\) is normal operator.
Chapter one  

some preliminary concepts

In this section we recall some basic concepts that we need later; we start with the definition of bilinear map.

**Definition**[1-1-1][1, p.116]

If $U, V, W$ are vector spaces, a mapping $\varphi : U \times V \to W$ is said to be bilinear in case the relations

\[
\varphi(x_1 + x_2, y) = \varphi(x_1, y) + \varphi(x_2, y) \\
\varphi(\lambda x, y) = \lambda \varphi(x, y) \\
\varphi(x, y_1 + y_2) = \varphi(x, y_1) + \varphi(x, y_2) \\
\varphi(x, \lambda y) = \lambda \varphi(x, y)
\]

$x_1, x_2, x \in U$  
y_1, y_2, y \in V  
$\lambda \in \mathbb{C}$

hold identically. If moreover $W = \mathbb{C}$, $\varphi$ is called a bilinear form on $U \times V$.

**Definition**[1-1-2][1, p.123]

If $U, V, W$ are vector spaces, a mapping $\varphi : U \times V \to W$ is said to be sesquilinear in case the relations.

\[
\varphi(x_1 + x_2, y) = \varphi(x_1, y) + \varphi(x_2, y) \\
\varphi(\lambda x, y) = \lambda \varphi(x, y) \\
\varphi(x, y_1 + y_2) = \varphi(x, y_1) + \varphi(x, y_2) \\
\varphi(x, \lambda y) = \overline{\lambda} \varphi(x, y)
\]

$x_1, x_2, x \in U$  
y_1, y_2, y \in V  
$\lambda \in \mathbb{C}$  
$\lambda \in \mathbb{C}$

hold identically. If moreover $W = \mathbb{C}$, $\varphi$ is called a sesquilinear form on $U \times V$. 

1. Preliminaries
Let $H$ be a Hilbert space and $f, g \in H$ then $f$ and $g$ are orthogonal if $\langle f, g \rangle = 0$ in symbols $f \perp g$. [7, p.7].

**Definition** [1-1-3] [7, p.14]

An orthonormal subset of a Hilbert space $H$ is a subset $S$ having the properties:
1. For $f$ in $S$, $\|f\| = 1$;
2. If $f, g \in S$ and $f \neq g$ then $f \perp g$

**Definition** [1-1-4] [1]

A set $S$ of vectors in a Hilbert space $H$ is said to be total in case the only vector $z$ of $H$ which is orthogonal to every vector of $S$ is the vector $z = 0$.

**Definition** [1-1-5] [16]

An operator $A$ on the Hilbert space is called bounded from below if there exist a positive number $\delta$ such that $\|Af\| \geq \delta \|f\|$ for every $f$ in $H$.

We recall now one of the basic notions related to operators which are defined on a complex Hilbert space $H$.

**Definition** [1-1-6] [16, p.41]

Let $H$ be a complex Hilbert space, and $A \in B(H)$.

The spectrum of $A$, $\sigma(A)$, is defined by

$$\sigma(A) = \{ \lambda \in \mathbb{C} | A - \lambda I \text{ is not invertible} \}$$

The spectral radius of an operator $A$, $r(A)$, is defined by

$$r(A) = \sup \{ \|\lambda\| : \lambda \in \sigma(A) \}$$

The approximate point spectrum of $A$, $\sigma_{ap}(A)$, is defined by
\[ \sigma_{\text{ap}}(A) = \{ \lambda \in \mathcal{F} | A - \lambda I \text{ is not bounded below} \} \]
The compression spectrum of \( A, \Gamma(A) \), is defined by

\[ \Gamma(A) = \{ \lambda \in \mathcal{F} | A - \lambda I \text{ has non dense range} \} \]

**Remarks** [1-1-7]

1. It is easily seen that \( \lambda \in \sigma_{\text{ap}}(A) \) if and only if there exists a sequence \( \{x_n\} \) of unit vectors such that \( \| (A - \lambda I)x_n \| \to 0 \) when \( n \to \infty \)
2. An important subset of \( \sigma_{\text{ap}}(A) \) is the point spectrum, \( \sigma_{p}(A) \), where
   \[ \sigma_{p}(A) = \{ \lambda \in \mathcal{F} | Af = \lambda f \text{ for some nonzero vector } f \} \]
3. \( \sigma(A) = \sigma_{\text{ap}}(A) \cup \Gamma(A) \)
4. The residual spectrum, \( \sigma_{r}(A) \), is defined by
   \[ \sigma_{r}(A) = \Gamma(A) - \sigma_{p}(A) \]

Another basic concept related to operators is the numerical range of an operator.

**Definition** [1-1-8] [16, p.112]

Let \( H \) be a complex Hilbert space, and let \( A \in B(H) \). the numerical range of an operator \( A, W(A) \), is defined by

\[ W(A) = \{ \langle Ax, x \rangle | x \in H, \| x \| = 1 \} \]

The numerical radius of an operator \( A, \omega(A) \) is defined by

\[ \omega(A) = \sup \{ |\lambda| : \lambda \in W(A) \} \]

**Let:** \( S \) be a nonempty subset of \( \mathcal{F}, S \) is said to be a convex set if for all \( x, y \in S, tx + (1 - t)y \in S \) where \( 0 \leq t \leq 1 \).

The convex hull of \( S \), denoted by \( \text{conv} S \) is the intersection of all convex sets containing \( S \) [12, p.414].

The following theorem is used later, for the proof see [16].
Chapter one

some preliminary concepts

Töplitz-Hausdorff theorem[1-1-9]

The numerical range of an operator is a convex subset of the complex plane.

The following proposition illustrates the relation between the point spectrum and the compression spectrum of an operator with its numerical range.

**Proposition**[1-1-10][16]

Let \( A \in B(H) \)

then, \( \sigma_p(A) \subseteq \overline{W(A)} \) and \( \Gamma(A) \subseteq \overline{W(A)} \)

Before the proof we need the following proposition.

**Proposition**[1-1-11]

Let \( A \) be an operator on \( H \),

1- \( W(A^*)=\overline{W(A)} \)

2. \( \sigma_p(A^*)=C(\Gamma(A)) \)

**Proof:-**

1-Let \( x \in H , \|x\|=1 \) then \( \langle A^*x,x \rangle = \langle x, Ax \rangle = \langle Ax, x \rangle \)

2- If, \( \lambda \in \sigma_p(A^*) \) then there exists \( x \in H , x \neq 0 \) such that, \( A^*x = \lambda x \) put \( H'=(A-\lambda I)H \), therefore to prove \( \overline{\lambda} \in \Gamma(A) \) it is enough to show that \( H' \) is not dense in \( H \), now for all \( y \in H \)

\[ \langle (A-\lambda I)y,x \rangle = \langle y,(A^*-\lambda I)x \rangle = 0 \]

Therefore there exists \( x \in H , x \neq 0 \) such that, \( x \in H'^\perp \), thus \( H' \) is not dense in \( H \) hence \( \overline{\lambda} \in \Gamma(A) \) and \( \sigma_p(A^*) \subseteq C(\Gamma(A)) \)

Conversely
If $\lambda \in C(\Gamma(A))$, then put $H' = (A - \lambda I)H$ therefore $H'$ is not dense in $H$, thus there exists $x \in H, x \neq 0$ such that 
\[ \langle (A - \lambda I)y, x \rangle = 0 \forall y \in H \]  
Hence 
\[ \langle y, (A^* - \lambda I)x \rangle = 0 \forall y \in H \]  
Therefore $(A^* - \lambda I)x = 0$ and it follows that $\sigma_p(A^*) = C(\Gamma(A))$.

**Proof** proposition [1-1-10]

If $\lambda \in \sigma_p(A)$ then there exists $x \in H, x \neq 0$ such that $Ax = \lambda x$.

Now 
\[ \lambda = \lambda \langle x/\|x\|, x/\|x\| \rangle = \langle \lambda x/\|x\|, x/\|x\| \rangle = \langle Ax/\|x\|, x/\|x\| \rangle \]
therefore $\lambda \in W(A)$ and $\sigma_n(A) \subseteq W(A)$.

By this fact and the last proposition finish the proof.

The following proposition gives a relation between the approximate point spectrum of an operator and its numerical range.

**Proposition** [1-1-12][16]

Let $A$ be an operator on $H$, then $\sigma_{ap}(A) \subseteq C(W(A))$.

**Proof:-**

If $\lambda \in \sigma_{ap}(A)$, then there exists a sequence $\{x_n\}$ of unite vectors in $H$ such that 
\[ \|(A - \lambda I)x_n\| \to 0 \text{ when } n \to \infty \]

Now 
\[ \|\langle Ax_n, x_n \rangle - \lambda \| = \|\langle (A - \lambda I)x_n, x_n \rangle \|
\leq \|(A - \lambda I)x_n\| \|x_n\|
= \|(A - \lambda I)x_n\| \to 0 \text{ when } n \to \infty \]
thus $\lim_{n \to \infty} \langle Ax_n, x_n \rangle = \lambda$ hence $\lambda \in C(W(A))$ then $\sigma_{ap}(A) \subseteq C(W(A))$. 

8
Chapter one some preliminary concepts

The following proposition illustrates the relation between the spectrum of an operator and its numerical range.

Since \( \sigma(A) = \sigma_r(A) \cup \sigma_{ap}(A) \) we get .

**Proposition** [1-1-13] [34,p.600]

For every operator \( A \) on a Hilbert space \( H \) \( \sigma(A) \subseteq C(W(A)) \) so that \( \text{conv}\sigma(A) \subseteq C(W(A)) \).

It is known that if \( A \) is self adjoint, then \( \sigma(A) \) is real [16] and \( \langle Ax, x \rangle \) is real for all \( x \in H \).

**Definition** [1-1-14] [30]

An operator \( A \) on a Hilbert space \( H \) is called hyponormal operator if \( A^*A - AA^* \geq 0 \) i.e. \( \langle (A^*A - AA^*)x, x \rangle \geq 0 \) for all \( x \in H \).

**Theorem** [1-1-15]

Let \( A \) be a hyponormal operator on \( H \) then \( \| A^n \| = \| A \|^n \) for all \( n \in N \)

**Definition** [1-1-16] [18, p.267]

Let \( A \) be an operator on \( H \), \( A \) is called normaloid if \( \| A \| = r(A) \).

since, \( r(A) = \lim_{n \to \infty} \| A^n \|^\frac{1}{n} \) [15] it is clear that every hyponormal operator is normaloid.

**Definition** [1-1-17][34, p539]

Let \( A \) be an operator on a Hilbert space \( H \), \( A \) belong to \( c_1 \) if \( C(W(A)) = \text{conv}\sigma(A) \) then \( A \) is called a convexoid operator ; to \( c_2 \) if \( \alpha A + \beta I \) is normaloid for all \( \alpha, \beta \in \mathbb{C} \).
Chapter one  some preliminary concepts

Lemma [1-1-18]

Every $c_2$ operator is $c_1$.

Proof:– see [34]

1.2. Tensor product of vector spaces

In this section we give the classical concept of the tensor product operation involving infinite dimensional vector spaces.

In the following remarks we study tensor product of vector space.

Remark [1-2-1] [33]

Let $H_1$ and $H_2$ be two vector spaces over the field the formal finite linear combinations of the pairs $(f, g)$ with $f \in H_1, g \in H_2$ is denoted by $F(H_1, H_2)$

$$F(H_1, H_2) = \left\{ \sum_{j=1}^{n} c_j (f_j, g_j) : c_j \in K, f_j \in H_1, g_j \in H_2, j = 1, 2, \ldots, n; n \in N \right\}$$

is a vector space

Remark [1-2-2] [33]

Let $N$ be the subspace of $F(H_1, H_2)$ spanned by the elements of the form

$$\sum_{j=1}^{n} a_j b_k (f_j, g_k) - 1 \times \left( \sum_{j=1}^{n} a_j f_j, \sum_{k=1}^{m} b_k g_k \right)$$

the quotient space

$$H_1 \otimes H_2 = F(H_1, H_2) / N$$

is called the algebraic tensor product of $H_1$ and $H_2$
the equivalence class from $H_1 \otimes H_2$ defined by $(f,g)$ will be denoted by $f \otimes g$, these elements are called simple tensors. Each element of $H_1 \otimes H_2$ is representable as a finite linear combination of simple tensors.

**Remark [1-2-3] [33]**

Linear combination of simple tensors is equal to zero if and only if it is a finite linear combination of elements of the form.

$$\sum_{j=1}^{n} \sum_{k=1}^{m} a_j b_k f_j \otimes g_j - \left( \sum_{j=1}^{n} a_j f_j \right) \otimes \left( \sum_{k=1}^{m} b_k g_k \right)$$

In particular, we have

$$\sum_{j=1}^{n} \sum_{k=1}^{m} a_j b_k f_j \otimes g_j = \left( \sum_{j=1}^{n} a_j f_j \right) \otimes \left( \sum_{k=1}^{m} b_k g_k \right)$$

Let $(H_1, \langle \cdot, \cdot \rangle_1)$ and $(H_2, \langle \cdot, \cdot \rangle_2)$ be Hilbert spaces.

**Remark [1-2-4]**

The mapping $S: F(H_1, H_2) \times F(H_1, H_2) \to K$ defined by

$$S \left( \sum_{j=1}^{n} c_j (f_j, g_j) \right) \sum_{k=1}^{m} c_k' (f'_k, g'_k) = \sum_{j=1}^{n} \sum_{k=1}^{m} c_j c'_k \langle f_j, f'_k \rangle_1 \langle g_j, g'_k \rangle_2$$

is a sesquilinear form on $F(H_1, H_2)$ see [32]

The following theorem appeared in [32] without proof, we give the proof.
Remark [1-2-5]

The mapping \( S : (H_1 \otimes H_2) \times (H_1 \otimes H_2) \to K \) defined by

\[
\left\{ \sum_{j=1}^{n} c_j (f_j \otimes g_j), \sum_{k=1}^{m} c_k' (f_k' \otimes g_k') \right\} = S \left\{ \sum_{j=1}^{n} c_j (f_j, g_j), \sum_{k=1}^{m} c_k' (f_k', g_k') \right\}
\]

is a sesquilinear form on \( H_1 \otimes H_2 \)

Proof:-

Let

\[
f = \sum_{j=1}^{n} c_j (f_j, g_j)
\]

\[
= \sum_{j=1}^{n} c_j \left( f_j^1, f_j^2, K, g_j^1, g_j^2, K \right)
\]

\[
g = \sum_{k=1}^{m} c_k' (f_k', g_k')
\]

\[
= \sum_{k=1}^{m} c_k' \left( f_k'^1, f_k'^2, K, g_k'^1, g_k'^2, K \right)
\]

\[
h = \sum_{t=1}^{l} c_t'' (f_t'', g_t'' , K)
\]

\[
= \sum_{t=1}^{l} c_t'' \left( f_t''^1, f_t''^2, K, g_t''^1, g_t''^2, K \right)
\]

\[
S(f + g, h) =
\]

\[
S \left\{ \sum_{j=1}^{n} c_j (f_j^1, f_j^2, K, g_j^1, g_j^2, K) + \sum_{k=1}^{m} c_k' (f_k'^1, f_k'^2, K, g_k'^1, g_k'^2, K) \right\} =
\]

\[
\sum_{t=1}^{l} c_t'' \left( f_t''^1, f_t''^2, K, g_t''^1, g_t''^2, K \right)
\]

\[
S \left\{ \sum_{j=1}^{n} \sum_{k=1}^{m} (c_j + c_k') (f_j^1 + f_k'^1, f_j^2 + f_k'^2, K, g_j^1 + g_k'^1, g_j^2 + g_k'^2) \right\} =
\]

\[
\sum_{t=1}^{l} c_t'' \left( f_t''^1, f_t''^2, K, g_t''^1, g_t''^2, K \right)
\]

\[
, f_t'', f_t, \left\{ g_j^1 + g_k'^1, g_j^2 + g_k'^2, K, g_t''^1, g_t''^2, K \right\} =
\]

12
\[
\sum_{j=1}^{n} \sum_{k=1}^{m} \left( c_j c_t + \overline{c} c_t \right) f_j^t \cdot f_t^g \cdot g_j^t \cdot g_t^g + f_k f_j g_j g_t + f_f f_t g_j g_t + f_f f_t g_j g_t + f_f f_t g_j g_t + f_f f_t g_j g_t + K \]

on the other hand

\[
S(f, h) + S(g, h) =
\]

\[
S( \sum_{j=1}^{n} c_j (f_j^t, f_j^g, K, g_j^t, g_j^g, K) \sum_{t=1}^{l} c_t (f_t^g, f_t^g, K, g_t^t, g_t^t, K) ) +
\]

\[
S( \sum_{k=1}^{m} c_k (f_k^t, f_k^g, K, g_k^t, g_k^g, K) \sum_{t=1}^{l} c_t (f_t^g, f_t^g, K, g_t^t, g_t^t, K) ) =
\]

\[
\sum_{j=1}^{n} \sum_{k=1}^{m} \sum_{t=1}^{l} (-c_j c_t) f_j^t f_k^t f_t^g g_j^t g_t^g + f_f f_t^g g_j^t g_t^g + f_f f_t^g g_j^t g_t^g + f_f f_t^g g_j^t g_t^g + f_f f_t^g g_j^t g_t^g + K
\]

\[
\sum_{j=1}^{n} \sum_{k=1}^{m} \sum_{t=1}^{l} (-c_j c_t) f_j^t f_k^t f_t^g g_j^t g_t^g + f_f f_t^g g_j^t g_t^g + f_f f_t^g g_j^t g_t^g + f_f f_t^g g_j^t g_t^g + f_f f_t^g g_j^t g_t^g + K + (c_j c_t)
\]

\[
f_j^t f_k^t + f_f f_t^g + K (g_j^t g_t^g + g_j^t g_t^g + K) =
\]

\[
\sum_{j=1}^{n} \sum_{k=1}^{m} \sum_{t=1}^{l} (-c_j c_t) f_j^t f_k^t f_t^g g_j^t g_t^g + f_f f_t^g g_j^t g_t^g + f_f f_t^g g_j^t g_t^g + f_f f_t^g g_j^t g_t^g + f_f f_t^g g_j^t g_t^g + K + f_f f_t^g g_j^t g_t^g + f_f f_t^g g_j^t g_t^g + f_f f_t^g g_j^t g_t^g + K
\]

hence \( S(f + g, h) = S(f, h) + S(g, h) \)

Similarly \( S(f, a) = a S(f, g) \)

hence \( S(\lambda f, g) = \lambda S(f, g) \)
Then it is clear that $H_1 \otimes H_2$ is pre-Hilbert space

**Remark** [1-2-6]

Note that the completion of this Pre–Hilbert space will be denoted by $H_1 \hat{\otimes} H_2$ and is called the tensor product of the Hilbert spaces $H_1$ and $H_2$.

The following theorem gives some properties of tensor product of Hilbert spaces

Denoted by $L(M)$ is the set of finite linear combinations of element of $M$

**Theorem** [1-2-7] [33]

Let $H_1$ and $H_2$ be two Hilbert spaces

(a) If $M_1$ and $M_2$ are total subsets of $H_1$ and $H_2$ respectively, then the set $\{f \otimes g : f \in M_1, g \in M_2\}$ is total in $H_1 \hat{\otimes} H_2$

(b) If $\{e_\alpha : \alpha \in A\}$ and $\{f_\beta : \beta \in B\}$ are orthonormal basis (orthonormal and total) of $H_1$ and $H_2$ respectively then $\{e_\alpha \otimes f_\beta : \alpha \in A, \beta \in B\}$ is an orthonormal basis of $H_1 \hat{\otimes} H_2$

**Proof:**

(a) Let $\sum_{j=1}^{n} f_j \otimes g_j \in H_1 \otimes H_2 \in C > 0$ for each $j = 1, 2, \ldots, n$

$\exists f'_j \in L(M_1)$ and $g'_j \in L(M_2)$ such that

$\|f_j - f'_j\| g_j \| C / 2n$ and $\|g_j - g'_j\| f_j \| C / 2n$

Then we have

$\|f_j \otimes g_j - f'_j \otimes g'_j\| = \|(f_j - f'_j) \otimes g_j + f'_j \otimes (g_j - g'_j)\| C / n$

$\sum_{j=1}^{n} f_j \otimes g_j \rightarrow \sum_{j=1}^{n} f'_j \otimes g'_j$

because $\sum_{j=1}^{n} f'_j \otimes g'_j \in L(M_1) \otimes L(M_2) = L\{f \otimes g, f \in M_1, g \in M_2\}$

Since $H_1 \otimes H_2$ is dense in $H_1 \hat{\otimes} H_2$ then $\sum_{j=1}^{n} f'_j \otimes g'_j \in H_1 \hat{\otimes} H_2$
(b) By part (a) the set \( \{ e_\alpha \otimes f_\beta : \alpha, \beta \in B \} \) is total in \( H_1 \otimes H_2 \) then it is enough to proof that \( \| e_\alpha \otimes f_\beta \| = 1 \)

\[ \langle e_\alpha \otimes f_\beta, e_\alpha' \otimes f_\beta' \rangle = \delta_{\alpha \alpha'} \delta_{\beta \beta'} \] such that \( \delta_{\alpha \alpha'} = \begin{cases} 1 & \alpha = \alpha' \\ 0 & \alpha \neq \alpha' \end{cases} \)

\[ \langle e_\alpha \otimes f_\beta, e_\alpha' \otimes f_\beta' \rangle = \langle e_\alpha, e_\alpha' \rangle \langle f_\beta, f_\beta' \rangle = \begin{cases} 0 & \text{If } \alpha \neq \alpha' \text{ or } \beta \neq \beta' \\ 1 & \text{If } \alpha = \alpha' \text{ and } \beta = \beta' \end{cases} \]

Hence \( e_\alpha \otimes f_\beta \) is orthonormal basis of \( H_1 \otimes H_2 \)

### 1.3. Tensor product of operators

In this section we study the properties of operators defined on the Hilbert space \( H_1 \otimes H_2 \), where each \( H_1 \) and \( H_2 \) is a Hilbert space and show that \( A \) can be written by \( A = A_1 \otimes A_2 \) where \( A_1 \in B(H_1) \) and \( A_2 \in B(H_2) \).

**Definition [1-3-1]**

Let \( H_1 \) and \( H_2 \) be two vector spaces over the field \( \mathbb{F} \) defined \( H_1 \times H_2 \) by \( H_1 \times H_2 = \{ (x, y) : x \in H_1, y \in H_2 \} \). Let \( F \) be the subgroup generated by:

\[
\begin{align*}
    a(x, y) &= (ax, y) \\
    a(x, y) &= (x, ay) \\
    (x + x', y) &= (x, y) + (x', y) \\
    (x, y + y') &= (x, y) + (x, y')
\end{align*}
\]

\( H_1 \times H_2 / F \) is the tensor product of \( H_1 \) and \( H_2 \) it is denoted by \( H_1 \otimes H_2 \). where \( (x, y) \mapsto (x \otimes y) \).

Let \( A_1 : H_1 \to H_1 \) and \( A_2 : H_2 \to H_2 \) define \( A_1 \times A_2 : H_1 \times H_2 \to H_1 \times H_2 \) \( (A_1 \times A_2)(x, y) = (A_1x, A_2y) \) it is clear that \( (A_1 \times A_2)(F) \subseteq F \) then \( A_1 \times A_2 \) induces a map \( A_1 \otimes A_2 : H_1 \otimes H_2 \to H_1 \otimes H_2 \) so that if \( x \in K \) and \( x' \in K' \):

\[
(A_1 \otimes A_2)(x \otimes y) = A_1x \otimes A_2x'.
\]
\textbf{Theorem [1-3-2] [26]}

Let $H_1$ and $H_2$ be two Hilbert spaces, $A$ be a bounded operator on the Hilbert tensor product of these two space, consider $F(u,v,w,z)=\langle u \otimes w, A(v \otimes z) \rangle$ if $A = A_1 \otimes A_2$ where $A_1 \in B(H_1)$ and $A_2 \in B(H_2)$ then $F(u,v,w,z)=\langle u, A_1 v \rangle \langle w, A_2 z \rangle$

\textbf{Proof:}

If $A = A_1 \otimes A_2$ and $F(u,v,w,z)=\langle u, A_1 v \rangle \langle w, A_2 z \rangle$ then $F(u,v,w,z)F(u',v',w',z')=\langle u, A_1 v \rangle \langle w, A_2 z \rangle \langle u', A_1 v' \rangle \langle w', A_2 z' \rangle$

$$= \langle u, A_1 v \rangle \langle w', A_2 z' \rangle \langle u', A_1 v' \rangle \langle w, A_2 z \rangle$$

$$= F(u,v,w',z')F(u',v',w,z)$$

For all $u,v,u',v' \in H_1$ and $w,z,w',z' \in H_2$

\textbf{Conversely:}

Suppose this equality holds, we may assume $F$ is not identically 0 .fixed some $u',v',w',z'$ such that $F(u',v',w',z')=1$ and consider the bounded sesquilinear form $\phi(w,z) \rightarrow F(u',v',w,z)$ . by the Riesz lemma [16] there is a bounded operator $A_2 \in H_2$ such that $\phi(w,z)=\langle w, A_2 z \rangle$

.similarly there is a bounded operator $A_1 \in H_1$ such that $\phi(u,v)=\langle u, A_1 v \rangle$ i.e. $F(u,v,w',z')=\langle u, A_1 v \rangle$

Then $F(u,v,w,z)= F(u,v,w,z)F(u',v',w',z')= F(u,v,w',z')F(u',v',w,z)$

i.e. $\langle u \otimes w, A(v \otimes z) \rangle = \langle u, A_1 v \rangle \langle w, A_2 z \rangle = \langle u \otimes w, A_1 v \otimes A_2 z \rangle$

For all $u,v,w,z$ since the linear span of $u \otimes v$ is dense in $H_1 \hat{\otimes} H_2$ conclude that $A(u \otimes v) = A_1 v \otimes A_2 z$ i.e. $A = A_1 \otimes A_2$

The next remarks describe some properties of tensor product of operator

The following remark appeared in [9] without proof ,we give a proof.
**Remark [1-3-3]**

Let $A_1$ and $A_2$ be two operators on $H_1$ and $H_2$ respectively then

$$(A_1 \otimes A_2)^* = A_1^* \otimes A_2^*$$

**Proof:**

Let $x, z \in H_1$ and $y, w \in H_2$

$$\langle (A_1 \otimes A_2)^*(x \otimes y), (z \otimes w) \rangle = \langle (x \otimes y), (A_1 \otimes A_2)(z \otimes w) \rangle$$

$$= \langle x, A_1 z \rangle \langle y, A_2 w \rangle$$

$$= \langle A_1^* x, z \rangle \langle A_2^* y, w \rangle$$

$$= \langle z, A_1^* x \rangle \langle w, A_2^* y \rangle$$

$$= \langle A_1^* \otimes A_2^* (x \otimes y), (z \otimes w) \rangle$$

Hence $(A_1 \otimes A_2)^* = A_1^* \otimes A_2^*$

The next remark appeared in [19] without proof, we give a proof

**Remark [1-3-4]**

Let $A_1$ and $A_2$ be two operators on $H_1$ and $H_2$ respectively then

$$(A_1 \otimes A_2)(A_1 \otimes A_2)^* = A_1 A_1^* \otimes A_2 A_2^*$$

**Proof:**

Let $x, z \in H_1$ and $y, w \in H_2$

$$\langle (A_1 \otimes A_2)(A_1 \otimes A_2)^*(x \otimes y), (z \otimes w) \rangle = \langle (A_1 \otimes A_2)^*(x \otimes y), (A_1 \otimes A_2)^* (z, w) \rangle$$

$$= \langle A_1^* x \otimes A_2^* y, A_1^* z \otimes A_2^* w \rangle$$

$$= \langle A_1^* x, A_1^* z \rangle \langle A_2^* y, A_2^* w \rangle$$

$$= \langle x, (A_1 A_1^*)^* z \rangle \langle y, (A_2 A_2^*)^* w \rangle$$

$$= \langle A_1 A_1^* x, z \rangle \langle A_2 A_2^* y, w \rangle$$

$$= \langle A_1 A_1^* \otimes A_2 A_2^* (x \otimes y), (z \otimes w) \rangle$$

Hence $(A_1 \otimes A_2)(A_1 \otimes A_2)^* = A_1 A_1^* \otimes A_2 A_2^*$. 
Chapter one  

Some preliminary concepts

In general we can prove the following remark

**Remark [1-3-5]**

For any \( A_1, A_2 \in H_1 \) and \( B_1, B_2 \in H_2 \) then

\[
(A_1 \otimes B_1)(A_2 \otimes B_2) = A_1 A_2 \otimes B_1 B_2
\]

**Proof:**

Let \( x, z \in H_1 \) and \( y, w \in H_2 \)

\[
\langle (A_1 \otimes B_1)(A_2 \otimes B_2) (x \otimes y), (z \otimes w) \rangle = \langle (A_2 \otimes B_2) (x \otimes y), (A_1 \otimes B_1)^* (z, w) \rangle
\]

\[
= \langle A_2 x \otimes B_2 y, A_1^* z \otimes B_1^* w \rangle
\]

\[
= \langle A_2 x, A_1^* z \rangle \langle B_2 y, B_1^* w \rangle
\]

\[
= \langle x, A_2^* A_1^* z \rangle \langle y, B_2^* B_1^* w \rangle
\]

\[
= \langle A_1 A_2 z, x \rangle \langle B_1 B_2 y, w \rangle
\]

\[
= \langle (A_1 \otimes B_1)(A_2 \otimes B_2) (x \otimes y), (z \otimes w) \rangle
\]

Hence \((A_1 \otimes B_1)(A_2 \otimes B_2) = A_1 A_2 \otimes B_1 B_2\).

We prove the following remark

**Remark [1-3-6]**

Let \( A_1 \) and \( A_2 \) be two operators on \( H_1 \) and \( H_2 \) respectively then

\[
(A_1 \otimes A_2)^n = A_1^n \otimes A_2^n \quad n \geq 1 \quad \text{and} \quad n \in \mathbb{N}
\]

**Proof:**

Let \( x, z \in H_1 \) and \( y, w \in H_2 \)

\[
\langle (A_1 \otimes A_2)^n (x \otimes y), (z \otimes w) \rangle = \langle (A_1 \otimes A_2)^{n-1} (x \otimes y), A_1^* A_2^* (z \otimes w) \rangle
\]

\[
= \langle (A_1 \otimes A_2)^{n-2} (x \otimes y), A_1^{*2} \otimes A_2^{*2} (z \otimes w) \rangle
\]

\[
= \langle x, A_1^{*n} z \rangle \langle y, A_2^{*n} w \rangle
\]

\[
= \langle A_1^n x, z \rangle \langle A_2^n y, w \rangle
\]

\[
= \langle A_1^n \otimes A_2^n (x \otimes y), (z \otimes w) \rangle
\]

Hence \((A_1 \otimes A_2)^n = A_1^n \otimes A_2^n\).
The following remark appeared in [9] without proof, we give a proof.

**Remark [1-3-7]**

Let $A_1$ and $A_2$ be two operators on $H_1$ and $H_2$ respectively then

$$\| (A_1 \otimes A_2) \| = \| A_1 \| \| A_2 \|$$

**Proof:**

Let $(x \otimes y) \in H_1 \otimes H_2$ and $x, y \neq 0$

$$\| A_1 \otimes A_2 \| = \sup \{ \| (A_1 \otimes A_2) (x \otimes y) \| \mid x \otimes y \in H_1 \otimes H_2 \}$$

$$= \sup \{ \| A_1 x \| \| A_2 y \| \mid x \in H_1, y \in H_2 \}$$

$$= \sup \{ \| A_1 x \| \| y \| \mid x \in H_1 \} \cdot \sup \{ \| A_2 y \| \| y \| \mid y \in H_2 \}$$

$$= \| A_1 \| \| A_2 \|$$

**Remark [1-3-8]**

Let $A_1$ and $A_2$ be two operators on $H_1$ and $H_2$ respectively then

$$(A_1 \otimes A_2) = 0 \text{ if } A_1 \text{ or } A_2 \text{ equal to zero.}$$

**Theorem [1-3-9] [34]**

For any operator $A_1$ and $A_2$ on $H$

$$\sigma_{ap} (A_1) \cdot \sigma_{ap} (A_2) \subseteq \sigma (A_1 \otimes A_2)$$

$$\left( \sigma (A_1) - \sigma_{ap} (A_1) \right) \left( \sigma (A_2) - \sigma_{ap} (A_2) \right) \subseteq \sigma (A_1 \otimes A_2)$$

**Proof:**

Let $\lambda \in \sigma_{ap} (A_1), \mu \in \sigma_{ap} (A_2)$

Then there exist sequences $\{x_n\}$ and $\{y_n\}$ of unit vectors such that

$$\| (A_1 - \lambda I) x_n \| \to 0 \text{ and } \| (A_2 - \mu I) y_n \| \to 0 \text{ when } n \to \infty$$

$$\| (A_1 \otimes A_2) (x_n \otimes y_n) - \lambda \mu (I \otimes I) (x_n \otimes y_n) \| =$$

$$\| A_1 (x_n) \otimes A_2 (y_n) - \lambda I (x_n) \otimes A_2 (y_n) + \lambda I (x_n) \otimes A_2 (y_n) - \lambda \mu I (x_n) \otimes I (y_n) \| =$$

$$\| (A_1 (x_n) - \lambda I (x_n)) \otimes A_2 (y_n) + \lambda I (x_n) \otimes (A_2 (y_n) - \mu I (y_n)) \| =$$
\[ \|(A_1 - \lambda I)x_n\| \leq \|A_2(\cdot y_n)\| + \|\lambda\|n\|(A_2 - \lambda I)y_n\| \to 0 \text{ when } n \to \infty \]

Hence
\[ \sigma_{ap}(A_1) \cdot \sigma_{ap}(A_2) \subseteq \sigma(A_1 \otimes A_2) \]

Now to proof the second assertion
Let \( \lambda \in \sigma(A_1) - \sigma_{ap}(A_1) \) and \( \mu \in \sigma(A_2) - \sigma_{ap}(A_2) \)

Then
\[
\begin{align*}
\lambda & \in \sigma_r(A_1) \quad \text{And} \quad \mu \in \sigma_r(A_2) \\
\lambda & \in \sigma_p(A_1^*) \subseteq \sigma_{ap}(A_1^*) \\
\lambda & \in \sigma_{ap}(A_1^*) \\
\lambda & \in \sigma_{ap}(A_1^*) \\
\lambda \mu & \in \sigma_{ap}(A_1^*) \sigma_{ap}(A_2^*) \subseteq \sigma(A_1^* \otimes A_2^*) \quad \text{by 1} \\
\lambda \mu & \in \sigma(A_1^* \otimes A_2^*) \text{ then } \lambda \mu \in \sigma(A_1 \otimes A_2). 
\end{align*}
\]

**Theorem [1-3-10]** [34, p621]

For any operators \( A_1 \) and \( A_2 \) on \( H \)
\[ \sigma(A_1 \otimes A_2) = \sigma(A_1) \cdot \sigma(A_2) \]

**Proof:-**

\[ \sigma(A_1) = \sigma(A_1 \otimes I) \quad \sigma(A_2) = \sigma(I \otimes A_2) \]

Then
\[ \sigma((A_1 \otimes I) \cdot (I \otimes A_2)) \subseteq \sigma(A_1 \otimes I) \cdot \sigma(I \otimes A_2) = \sigma(A_1) \cdot \sigma(A_2) \]

(\( \Leftarrow \)

We have only to treat the case of
\[ \lambda \in \sigma_{ap}(A_1) \quad \text{and} \quad \mu \in \sigma(A_2) - \sigma_{ap}(A_2) \]

or
\[ \lambda \in \sigma(A_1) - \sigma_{ap}(A_1) \quad \text{and} \quad \mu \in \sigma_{ap}(A_2) \]

Suppose
\[ \lambda \in \sigma_{ap}(A_1) \quad \text{and} \quad \mu \in \sigma(A_2) - \sigma_{ap}(A_2) \]

If \( \lambda = 0 \), \( \lambda \mu = \lambda \mu_0 = 0 \) for any point \( \mu_0 \in \sigma_{ap}(A_2) \)

So that
\[ \lambda \mu \in \sigma_{ap}(A_1) \cdot \sigma_{ap}(A_2) \subseteq \sigma(A_1 \otimes A_2) \text{ then } \lambda \mu \in \sigma(A_1 \otimes A_2) \]

The same by \( \mu = 0 \)
We may assume that \( \lambda \neq 0 \)

- **Case (1)** \( \bar{\lambda} \in \sigma_{ap}(A^*_1) \) since \( \bar{\mu} \in \sigma_p(A^*_2) \subseteq \sigma_{ap}(A^*_2) \) then
  \[ \bar{\lambda} \bar{\mu} \in \sigma_{ap}(A^*_1) \sigma_{ap}(A^*_2) \subseteq \sigma(A^*_1 \otimes A^*_2) \]

- **Case (2)** \( \bar{\lambda} \in \sigma_{ap}(A^*_1) \) and \( \lambda \in \sigma(A_1) - \sigma_{ap}(A_1) \) since for each operator \( A_i \) the approximate point spectrum \( \sigma_{ap}(A_i) \) is closed then \( \sigma(A_i) - \sigma_{ap}(A_i) \) is open and \( \sigma_{ap}(A_i) \) contains the boundary of \( \sigma(A_i) \)

Since \( \sigma(A^*_1) - \sigma_{ap}(A^*_1) \) and \( \sigma(A^*_2) - \sigma_{ap}(A^*_2) \) are open

\( \tau \bar{\lambda} \in \sigma(A^*_1) - \sigma_{ap}(A^*_1) \) and \( \mu/\tau \in \sigma(A^*_2) - \sigma_{ap}(A^*_2) \) for \( \tau \geq 1 \)

Sufficiently close to 1

\( \exists \tau_0 \) such that \( \tau_0 \bar{\lambda} \in \sigma_{ap}(A^*_1) \) or \( \mu/\tau_0 \not\in \sigma_{ap}(A^*_2) \) for \( 1 \leq \tau < \tau_0 \) suppose \( \tau \bar{\lambda} \in \sigma_{ap}(A_i) \) and \( \mu/\tau_0 \not\in \sigma_{ap}(A_2) \) then

\( \bar{\lambda} \bar{\mu} = (\tau_0/\bar{\lambda})(\mu/\tau_0) \in \sigma(A^*_1 \otimes A^*_2) \) for \( \bar{\mu}/\tau_0 \in \sigma_p(A^*_2) \)

hence \( \lambda \mu \in \sigma(A_1 \otimes A_2) \)

The case of \( \tau_0 \bar{\lambda} \in \sigma_{ap}(A^*_1) \) and \( \mu/\tau_0 \in \sigma_{ap}(A_2) \) is treated similarly

finally suppose that \( \tau_0 \bar{\lambda} \in \sigma_{ap}(A^*_1) \) and \( \mu/\tau_0 \in \sigma_{ap}(A_2) \)

Let \( \tau_n \) be a sequence such that \( 1 \leq \tau_n < \tau_0 \), \( \tau_n \to \tau_0 \) when \( n \to \infty \) then

for each \( n \), \( \tau_n \bar{\lambda} \in \sigma_{ap}(A^*_1) \) so that \( \tau_n \lambda \in \sigma_p(A_1) \) by theorem (1-3-8)

\( (\tau_n \lambda)(\mu/\tau_0) \in \sigma(A_1 \otimes A_2) \) Letting \( \tau_n \to \tau_0 \) we have \( \lambda \mu \in \sigma(A_1 \otimes A_2) \)

because \( \sigma(A_1 \otimes A_2) \) is close

The above theorem gives the following corollary

**Corollary** [1-3-11]

Let \( A_1 \) and \( A_2 \) be two operators on Hilbert space \( H \) \( A_1 \otimes A_2 \) is invertible if and only if \( A_1 \) and \( A_2 \) are, and the inverse of \( A_1 \otimes A_2 \) is \( A_i^{-1} \otimes A_2^{-1} \)

**Proof:**

Since \( A_1 \otimes A_2 \) is invertible then

\( 0 \in \sigma(A_1 \otimes A_2) \Leftrightarrow 0 \in \sigma(A_1) \) and \( 0 \in \sigma(A_2) \) then \( A_1 \) and \( A_2 \) are invertible
If \( A_1 \) and \( A_2 \) are invertible
\[
A_1A_1^{-1} = I = A_1^{-1}A_1 \\
A_2A_2^{-1} = I = A_2^{-1}A_2
\]
\[
(A_i \otimes A_2)(A_i^{-1} \otimes A_2^{-1}) = A_iA_i^{-1} \otimes A_2A_2^{-1} = I \otimes I
\]
the inverse is unique then
\[
(A_i \otimes A_2)^{-1} = A_i^{-1} \otimes A_2^{-1}
\]

The following proposition gives a relation between the closure of convex hull of numerical range of two operators \( A_1 \) and \( A_2 \) with the closure of the numerical range of the tensor product of \( A_1 \) and \( A_2 \).

**Proposition** [1-3-12] [34][17]

For any operators \( A_1 \) and \( A_2 \) on \( H \)
\[
C(\text{conv})(W(A_1) \cdot W(A_2)) \subseteq C(W(A_1 \otimes A_2))
\]

**Proof:**

Let \( \lambda \in W(A_1) \) and \( \mu \in W(A_2) \), then there exist unite vectors \( x, y \in H \) such that
\[
\lambda = \langle A_1x, x \rangle, \mu = \langle A_2y, y \rangle
\]
\[
\lambda \mu \in \langle A_1x, x \rangle \langle A_2y, y \rangle = \langle (A_1 \otimes A_2)(x \otimes y), (x \otimes y) \rangle
\]
Hence \( \lambda \mu \in W(A_1 \otimes A_2) \)

Since \( C(W(A_1 \otimes A_2)) \) is closed and convex we have
\[
C(\text{conv})(W(A_1) \cdot W(A_2)) \subseteq C(W(A_1 \otimes A_2))
\]

The converse of this proposition is not true in general. But in the following theorem it give the condition that makes the converse true

**Theorem** [1-3-13] [34]

Let \( A_1 \) and \( A_2 \) be two operators on a Hilbert space \( H \) then
\[
C(W(A_1 \otimes A_2)) = C(\text{conv})(W(A_1) \cdot W(A_2))
\]
if one of the following condition holds
1. If \( A_1 \otimes A_2 \) belong to \( c_1 \)
2. if \( A_1 \) and \( A_2 \) are hyponormal operators
Proof: 1- Since \(\text{conv}\sigma(A_1) \subseteq C(W(A_1))\) and \(\text{conv}\sigma(A_2) \subseteq C(W(A_2))\)

\[
\text{conv}(\sigma(A_1 \otimes A_2)) = \text{conv}(\sigma(A_1) \cdot \sigma(A_2))
\]

\[
= \text{conv}(\text{conv}\sigma(A_1) \cdot \text{conv}\sigma(A_2))
\]

\[
\subseteq \text{conv}(C(W(A_1)W(A_2)))
\]

\[
= C(\text{conv}(W(A_1) \cdot W(A_2)))
\]

\[
\subseteq C(W(A_1 \otimes A_2)) = \text{conv}(\sigma(A_1 \otimes A_2))
\]

Then \(C(W(A_1 \otimes A_2)) = C(\text{conv}(W(A_1) \cdot W(A_2)))\)

2- If \(A_1\) and \(A_2\) are hyponormal operators then

\[
A_1^*A_1 - A_1A_1^* \geq 0 \quad \text{and} \quad A_2^*A_2 - A_2A_2^* \geq 0
\]

we have

\[
\begin{align*}
(A_1 \otimes A_2)^* & (A_1 \otimes A_2) - (A_1 \otimes A_2)(A_1 \otimes A_2)^* = \\
(A_1^*A_1 - A_1A_1^*) \otimes A_2^*A_2 + A_1A_1^* \otimes (A_2^*A_2 - A_2A_2^*) \geq 0
\end{align*}
\]

Thus

\[
A_1 \otimes A_2\] is also hyponormal, so that for any complex numbers \(\alpha\) and \(\beta\)

\[
(\alpha(A_1 \otimes A_2) + \beta(I \otimes I))^*(\alpha(A_1 \otimes A_2) + \beta(I \otimes I)) - (\alpha(A_1 \otimes A_2) + \beta(I \otimes I))
\]

\[
(\alpha(A_1 \otimes A_2) + \beta(I \otimes I))^* = (\overline{\alpha}(A_1 \otimes A_2)^* + \overline{\beta}(I \otimes I))(\alpha(A_1 \otimes A_2) + \beta(I \otimes I))
\]

\[
(\alpha(A_1 \otimes A_2) + \beta(I \otimes I))(\overline{\alpha}(A_1 \otimes A_2)^* + \overline{\beta}(I \otimes I))
\]

\[
= |\alpha|^2 (A_1 \otimes A_2)^* (A_1 \otimes A_2) + \overline{\alpha} \beta (A_1 \otimes A_2)^* + \overline{\beta} \alpha (A_1 \otimes A_2)^* + |\beta|^2 (I \otimes I) -
\]

\[
= |\alpha|^2 ((A_1 \otimes A_2)^*(A_1 \otimes A_2) - (A_1 \otimes A_2)(A_1 \otimes A_2)^*) \geq 0
\]

Hence \(\alpha(A_1 \otimes A_2) + \beta(A_1 \otimes A_2)\) is hyponormal, and hence is normaloid

For each pair \(\alpha, \beta\) so that \((A_1 \otimes A_2) \in c_2\) By lemma (1-1-17) \((A_1 \otimes A_2) \in c_1\), then by 1

\[
C(W(A_1 \otimes A_2)) = C(\text{conv}(W(A_1) \cdot W(A_2)))
\]
Similarly if 
\( A_2 A_2^* A_2, A_2, I_2, A_2 A_2^*, A_2^*, A_2^2 A_2, A_2 A_2^*, A_2^* A_2 A_2^* \) are linearly dependent then
\[ I_1 = A_1^* A_1 = A_1 A_1^* A_1 = A_1^* = A_1 A_1^{*2} = A_1 = A_1 A_1^* = A_1 A_1^* = A_1^* A_1 A_1^* = 0 \]
Contradiction
hence \( A_2 A_2^* A_2, A_2, I_2, A_2 A_2^*, A_2^*, A_2^2 A_2, A_2 A_2^*, A_2^* A_2 A_2^* \) are linearly independent
then it is clear \( A_2 \) is normal operator
\( \iff \) since \( A_1 \) and \( A_2 \) are normal operators then \( A_1 A_1^* = A_1^* A_1 \) and
\( A_2 A_2^* = A_2^* A_2 \) it is easily seen that \( A_1 \otimes I_2 + I_1 \otimes A_2 \) is \( \theta \text{-- operator} \)

The following theorem gives the relation between \( \theta \text{-- operator} \) tensor product and quasinormal operators.

The following theorem appeared in [19] without proof and we give a proof.

**Theorem [2-2-18]**

\( A_1 \otimes A_2 \otimes L \otimes A_n \) is a \( \theta \text{-- operator} \) if and only if one of the following conditions holds:

1- \( A_j = 0 \) for some \( j \);
2- \( A_j \)'s are all quasinormal;
3-There exist real numbers \( a_1, a_2, K, a_n \) with \( a_1 + a_2 + K + a_n = 0 \) and some \( j \) such that \( e^{ia_j} A_j \) is \( \theta \text{-- operator} \) while \( e^{ia_k} A_k \) \((1 \leq k \leq n, k \neq j)\) are all self-adjoint operators.

**Proof:**

By induction, it suffices to show that \( A_1 \otimes A_2 \) is \( \theta \text{-- operator} \) operator if and only if 1 or 2 or 3 hold

Let \( A_1 = 0 \) and \( A_2 \neq 0 \)
(A_1 \otimes A_2)^* (A_1 \otimes A_2) \left( (A_1 \otimes A_2) + (A_1 \otimes A_2)^* \right) = \left( (A_1 \otimes A_2) + (A_1 \otimes A_2)^* \right) (A_1 \otimes A_2)^* (A_1 \otimes A_2)
A_1^* A_1^2 \otimes A_2^* A_2^2 + A_1^* A_1 A_1^* \otimes A_2 A_2^* A_2 - A_1 A_1^* A_2 A_2^* A_2 - A_1^2 A_1 \otimes A_2^2 A_2 =
0 \otimes A_2^* A_2^2 + 0 \otimes A_2^* A_2^2 - 0 \otimes A_2^* A_2^2 = 0
Hence \((A_1 \otimes A_2)\) is \(\theta - \text{operator}\) operator.

2- The first let \(A_1 \neq 0\) and \(A_2 \neq 0\)
Suppose that \(A_1 \otimes A_2\) is \(\theta - \text{operator}\)
A_1^* A_1 \otimes A_2^2 A_2 + A_1^* A_1 A_1^* \otimes A_2 A_2^* A_2 - A_1 A_1^* A_2 A_2^* A_2 - A_1^2 A_1 \otimes A_2^2 A_2 = 0
If \(A_1^* A_1 A_1^*\) and \(A_1^2 A_1\) are linearly independent then \(A_2^* A_2^* = A_2^* A_2 = 0\)
hence \(A_2 = 0\) contradiction
then \(A_1^* A_1 A_1^*\) and \(A_1^2 A_1\) are linearly dependent then \(A_1^* A_1 A_1^* = r A_1^2 A_1\)

Now
If \(A_1^2 A_2 A_2^*\) and \(A_2^2 A_2\) are linearly independent then \(A_1^* A_1 A_1^* = A_1^2 A_1 = 0\)
hence \(A_1 = 0\) contradiction

Then \(A_2^* A_2 A_2^*\) and \(A_2^2 A_2\) are linearly dependent then
\(A_2^* A_2 A_2^* = r A_2^2 A_2\)
Then \(r = 1\)
\(\Leftarrow\) Suppose that \(A_1\) and \(A_2\) are quasinormal operator it is easy to see that
\(A_1 \otimes A_2\) is \(\theta - \text{operator}\)

3- Let \(e^{i a_1} A_1\) is \(\theta - \text{operator}\)
\(\left( e^{-i a_1} A_1^* \right) \left( e^{i a_1} A_1 \right) \left( e^{i a_1} A_1 + e^{-i a_1} A_1^* \right) = \left( e^{i a_1} A_1 + e^{-i a_1} A_1^* \right) \left( e^{-i a_1} A_1^* \right) \left( e^{i a_1} A_1 \right)\)
\(e^{i a_1} A_1^* A_1^2 + e^{-i a_1} A_1^* A_1 A_1^* = e^{i a_1} A_1 A_1^* A_1 + e^{-i a_1} A_1^2 A_1\)
and let \(e^{i a_2} A_2\) is self-adjoint operator then \(e^{i a_2} A_2 = e^{-i a_2} A_2^*\)
to prove \(A_1 \otimes A_2\) is \(\theta - \text{operator}\)
\(A_1^* A_1^2 \otimes A_2 A_2^2 + A_1^* A_1 A_1^* \otimes A_2 A_2^* A_2^* - A_1 A_1^* A_1 \otimes A_2 A_2^* A_2 - A_1^2 A_1 \otimes A_2^2 A_2 =
\)
\(e^{i a_1} A_1^* A_1^2 \otimes e^{i a_2} A_2 A_2^2 + e^{i a_1} A_1 A_1^* \otimes e^{-i a_2} A_2 A_2^* A_2^* -
\)
\(e^{-i a_1} A_1 A_1^* \otimes e^{i a_2} A_2 A_2^* A_2 - e^{i a_1} A_1^2 \otimes e^{-i a_2} A_2^2 A_2 =\)
\[ e^{ia_1} A_1^* A_1^2 \otimes A_2^* e^{ia_2} A_2^2 + e^{-ia_1} A_1 A_1^* \otimes e^{ia_2} A_2 A_2^2 - \]
\[ e^{ia_1} A_1 A_1^* \otimes e^{-ia_2} A_2^* A_2 - e^{ia_1} A_1^* A_1 \otimes e^{-ia_2} A_2^* A_2 = \]
\[ \left( e^{ia_1} A_1^2 + e^{-ia_1} A_1 A_1^* - e^{ia_1} A_1 A_1^* A_1 + e^{-ia_1} A_1^* A_1 \right) \otimes e^{-ia_2} A_2^* A_2 = \]
\[ 0 \otimes e^{-ia_2} A_2^* A_2 = 0 \]

Hence \( A_1 \otimes A_2 \) is \( \theta \)-operator

\( \Leftarrow \) By proof 2 it is easily seen that \( A_1 = A_1^* \) and \( A_2 = A_2^* \) then by theorem (2-2-14) the proof is done.
CHAPTER THREE

Some properties of operators that are invariant under tensor product part II

For a fixed $A_1$ and $A_2$ in $B(H)$ the generalized derivations operator $\delta_{A_1,A_2}(X): B(H) \to B(H)$ is defined by: -

$$\delta_{A_1,A_2}(X) = A_1X - XA_2$$

for all $X \in B(H)$

and the elementary operator $\tau_{A_1,A_2}(X): B(H) \to B(H)$ be defined by: -

$$\tau_{A_1,A_2}(X) = A_1XA_2$$

for all $X \in B(H)$

In this chapter we study the compactness and the strongly stable operators $A_1 \otimes A_2$ we also study the identification between $A_1 \otimes A_2$ and $\tau_{A_1,A_2}$

This chapter consists of three sections

In § 3-1 we study the compactness of tensor product

In § 3-2 we study the identification between tensor product and elementary operators

In § 3-3 we discuss the relation between tensor product and strong stability
Let $H$ be separable complex Hilbert space, we recall that an operator $A \in B(H)$ is called compact operator if for every bounded sequence of vectors $\{x_n\}$, the sequence $A\{x_n\}$ has a convergent subsequence.

In this section we study the compactness of tensor product

Let $A_1, A_2, \ldots, A_n \in B(H_1)$ and $B_1, B_2, \ldots, B_n \in B(H_2)$

**Theorem** [3-1-1][19]

Let $A_{ij} \in B(H_1)$ be nonzero. If $\Phi = \sum_{i=1}^{m} A_{ij} \otimes A_{kj} \otimes K \otimes A_{nj}$ is compact and if, for some $r$, $A_{r_1}, A_{r_2}, \ldots, A_{r_m}$ are linearly independent then $A_{ij}$'s, with $i \neq r$, are all compact operators.

**Proof**

Without loss of generality we may assume $r = 1$ i.e. $A_{11}, A_{12}, \ldots, A_{1m}$ are linearly independent. Denote $T_j = A_{2j} \otimes A_{3j} \otimes K \otimes A_{nj}$ on $K = H_2 \otimes H_3 \otimes K \otimes H_n$ and $A_j = A_{1j}$, then $\Phi = A_1 \otimes T_1 + K + A_m \otimes T_m$ is compact. Since $A_1$ is not a linear combination of $A_2, \ldots, A_m$ there are vectors $x_1, K, x_s$ and $y_1, K, y_s$ such that

$$\sum_{k=1}^{s} \langle A_j x_k, y_k \rangle = \begin{cases} 1 & j = 1 \\ 0 & j \neq 1 \end{cases}$$

Thus, for $f, g \in K$ we have
\[ \langle T_1 f, g \rangle = \sum_{k=1}^{\infty} \langle A_j x_k, y_k \rangle \langle T_1 f, g \rangle = \langle \Phi (x_k \otimes f), (y_k \otimes g) \rangle \quad \ldots (3-1) \]

If \( f_t \) is an orthonormal sequence of vectors in \( K \) then \( \{ x_k \otimes f_t \} \) is a bounded orthogonal sequence in \( H_1 \otimes K \) for every \( k = 1, 2, K, s \) since \( \Phi \) is compact, we have \( \lim_{t \to \infty} \| \Phi (x_k \otimes f_t) \| = 0 \) for each \( k \)

Let \( g_t = T_1 f_t \) then by (3-1)
\[ \langle T_1 f_t, g_t \rangle = \| T_1 f_t \|^2 \]
\[ \lim_{t \to \infty} \| T_1 f_t \|^2 = \lim_{t \to \infty} \sum_{k=1}^{\infty} \langle \Phi (x_k \otimes f_t), (y_k \otimes g_t) \rangle = 0 \]

this implies that \( T_1 \) is compact operator, similarly \( T_2, K, T_m \) are compact. then \( T_j = A_2 j \otimes A_3 j \otimes K \otimes A_{nj} \) will imply compactness of each \( A_{ij} \) with \( i \geq 1 \).

The following theorem appeared in [19] without proof and we give a proof.

**Theorem (3-1-2)**

Let \( \Phi = A_1 \otimes B_1 + K + A_n \otimes B_n \in B\big( H_1 \otimes H_2 \big) \) if \( \{ A_1, A_2, K, A_n \} \) and \( \{ B_1, B_2, K, B_n \} \) are linearly independent, then \( \Phi \) is compact if and only if \( A_i \)'s and \( B_i \)'s are all compact.

**Proof:**

Suppose that \( \Phi \) is compact \( A_i \)'s are linearly independent then there are vectors \( x_1, K, x_s \) and \( y_1, K, y_s \) such that
\[ \sum_{k=1}^{s} \langle A_i x_k, y_k \rangle = \begin{bmatrix} 1 & i = 1 \\ 0 & i \neq 1 \end{bmatrix} \]

\[ \langle B_1 f, g \rangle = \sum_{k=1}^{s} \langle A_i x_k, y_k \rangle \langle B_1 f, g \rangle = \sum_{k=1}^{s} \sum_{i=1}^{n} \langle \Phi (x_k \otimes f), (y_k \otimes g) \rangle \]
Chapter three

Some properties of operators that are invariant under tensor product: part II

If $f_t$ is an orthonormal sequence of vectors in $H_2$ then \( \{x_k \otimes f_t\} \) is a bounded orthogonal sequence in $H_1 \hat{\otimes} H_2$, since $\Phi$ is compact then

\[
\lim_{t \to \infty} \|\Phi (x_k \otimes f_t)\| = 0 \quad \text{for each } k
\]

Let $g_t = B_1 f_t$ then

\[
\lim_{t \to \infty} \|B_1 f_t\|^2 = \lim_{t \to \infty} \sum_{k=1}^{s} \langle \Phi (x_k \otimes f_t), (y_k \otimes g_t) \rangle = 0
\]

Then $B_1$ is compact. Similarly $B_1, B_2, K, B_n$ are compact.

Since $B_i$'s are linearly independent, then there exist $x_1', K, x_s'$ and $y_1', K, y_s'$ such that

\[
\sum_{k=1}^{s} \langle x_k', B_i y_k' \rangle = \begin{cases} 1 & i = 1 \\ 0 & i \neq 1 \end{cases}
\]

\[
\langle f, A_i g \rangle = \sum_{k=1}^{s} \langle x_k', B_i y_k' \rangle \langle f, A_i g \rangle = \sum_{k=1}^{s} \langle x_k \otimes f, \Phi^* (y_k' \otimes g) \rangle
\]

Hence $\Phi^*$ is compact.

Then if $g_t$ is an orthonormal sequence of vector in $H_2$ then $\{y_k' \otimes g_t\}$ is a bounded orthogonal sequence in $H_1 \hat{\otimes} H_2$ since $\Phi^*$ is compact.

\[
\lim_{t \to \infty} \|\Phi^* (y_k' \otimes g_t)\| = 0 \quad \forall k
\]

Suppose $f_t = A_i g_t$ then

\[
\lim_{t \to \infty} \|A_i g_t\|^2 = \lim_{t \to \infty} \sum_{k=1}^{s} \langle x_k \otimes f_t, \Phi^* (y_k' \otimes g_t') \rangle = 0
\]

hence $A_i$ is compact similarly $A_2, A_3, K, A_n$ are compact.

Conversely

Suppose $A_i$'s and $B_i$'s are all compact then

\[
\lim_{t \to \infty} \|A_i f_t\| = 0 \quad \text{and} \quad \lim_{t \to \infty} \|B_i g_t\| = 0
\]

\[
\lim_{t \to \infty} \|\Phi (f_t \otimes g_t)\| \leq \lim_{t \to \infty} \|(A_i \otimes B_i) (f_t \otimes g_t)\| + K + \lim_{t \to \infty} \|(A_n \otimes B_n) (f_t \otimes g_t)\| = 0
\]

Hence $\Phi$ is compact.

The following theorem appeared in [19], we state it without proof.

...
Chapter three  

some properties of operators that are invariant under tensor product part II

**Theorem [3-1-3]**

Let  \( \Phi = A_1 \otimes B_1 + K + A_n \otimes B_n \in B(H_1 \hat{\otimes} H_2) \) is compact if and only if there exist compact operators  
\( E_1, K, E_r \in \text{span}\{A_1, K, A_n\} \) and  \( F_1, K, F_r \in \text{span}\{B_1, K, B_n\} \) such that  
\( \Phi = E_1 \otimes F_1 + K + E_r \otimes F_r \)

The following corollary appeared in [18] without prove, we give a proof

**Corollary [3-1-4]**

\( \Phi = A_1 \otimes B_1 + A_2 \otimes B_2 \) is compact if and only if one of the following conditions holds:-

1- there are operators  \( E, F \) and scalars  \( a, b, c, d \) with  \( ab + cd = 0 \) such that  
\( A_1 = aE, B_1 = bF, A_2 = cE, B_2 = dF; \)

2-  \( A_1, A_2, B_1 \) and  \( B_2 \) are all compact ;

3-  \( A_1 \) or  \( (B_1) \) is not compact , there is a scalar  \( r \) such that  \( B_1 = rB_2 \) or  \( (A_1 = rA_2) \), while  \( B_2 \) and  \( rA_1 + A_2 \) or  \( (A_2 \) and  \( rB_1 + B_2 \) ) are compact ;

4-  \( A_2 \) or  \( (B_2) \) is not compact , there is a scalar  \( r \) such that  \( B_2 = rB_1 \) or  \( (A_2 = rA_1) \), while  \( B_1 \) and  \( A_1 + rA_2 \) (or  \( A_1 \) and  \( B_1 + rB_2 \) ) are compact .

**Proof:** -

Suppose that  \( A_1 = aE, B_1 = bF, A_2 = cE, B_2 = dF \)

Let  \( (x_i \otimes y_i) \) be subsequence in  \( H_1 \hat{\otimes} H_2 \)

\[
\lim_{t \to \infty} \|(A_1 \otimes B_1 + A_2 \otimes B_2)(x_i \otimes y_i)\|
= \lim_{t \to \infty} \|abE \otimes F + (cd)E \otimes F(x_i \otimes y_i)\|
= \lim_{t \to \infty} \|ab + cd(E \otimes F)(x_i \otimes y_i)\| = 0
\]

Hence  \( \Phi \) is compact

Conversely:
Chapter three

some properties of operators that are invariant under tensor product part II

Since $\Phi$ is compact by theorem (3-1-3) there exists a compact operator $A_i = aE, B_i = bF, A_2 = cE, B_2 = dF$ let subsequence $(x_t \otimes y_t)$ in $H_1 \hat{\otimes} H_2$

$$\lim_{t \to \infty} (A_1 \otimes B_1 + A_2 \otimes B_2) (x_t \otimes y_t) = \lim_{t \to \infty} \left\| (ab)E \otimes F + (cd)E \otimes F (x_t \otimes y_t) \right\|$$

$$= \lim_{t \to \infty} \left\| (ab + cd)(E \otimes F)(x_t \otimes y_t) \right\|$$

$$= \lim_{t \to \infty} \left\| ab + cd \right\| E \otimes F (x_t \otimes y_t) = 0$$

If $\lim_{t \to \infty} E \otimes F (x_t \otimes y_t) = 0$ either $\lim_{t \to \infty} E x_t = 0$ or $\lim_{t \to \infty} F y_t = 0$ contradiction then $|ab + cd| = 0$

2-If $\Phi$ is compact then

If $\{A_1, A_2\}$ are linearly independent and $\{B_1, B_2\}$ are linearly independent then the proof immediately following from theorem (3-1-2)

If $\{A_1, A_2\}$ are linearly dependent and $\{B_1, B_2\}$ are linearly dependent then the proof immediately following from theorem (3-1-3)

3- Since $A_1$ is not compact, and $\Phi$ is compact, then $B_1, B_2$ are linearly dependent, then $B_1 = rB_2$

$$\Phi = A_1 \otimes rB_2 + A_2 \otimes B_2 = rA_1 \otimes B_2 + A_2 \otimes B_2 = (rA_1 + A_2) \otimes B_2$$

$rA_1 + A_2$ and $B_2$ are compact.

$\iff$ Suppose that $rA_1 + A_2$ and $B_2$ are compact

Then

$$\left( rA_1 + A_2 \right) \otimes B_2 = rA_1 \otimes B_2 + A_2 \otimes B_2 = A_1 \otimes rB_2 + A_2 \otimes B_2 = \Phi$$

Hence $\Phi$ is compact

4- $A_2$ is not compact, then $B_1, B_2$ are linearly dependent $B_2 = rB_1$

$A_1 \otimes B_1 + A_2 \otimes rB_1 = \left( A_1 + rA_2 \right) \otimes B_1 = \Phi$ Since $\Phi$ is compact $rA_1 + A_2$ and $B_1$ are compact.

$\iff$ Suppose $A_1 + rA_2$ and $B_1$ are compact.

$$(A_1 + rA_2) \otimes B_1 = A_1 \otimes B_1 + rA_2 \otimes B_1 = A_1 \otimes B_1 + A_2 \otimes rB_1$$

$$= A_1 \otimes B_1 + A_2 \otimes B_2 = \Phi$$

56
Chapter three  

The following corollary appeared in [19] without prove and we give the proof

**Corollary [3-1-5]**

\[ A_1 \otimes I_2 + I_1 \otimes A_2 \]  is compact if and only if  \( A_1 = rI_1 \)  and \( A_2 = -rI_2 \)  for some scalar \( r \)

**Proof:**-

Since \( I_1 \) and \( I_2 \) are not compact in infinite dimensional then \( \{ A_1, I_1 \} \) and \( \{ A_2, I_2 \} \) are linearly dependent hence \( A_1 = aI_1 \) and \( A_2 = bI_2 \) \( a, b \in K \)

let \( a = -b = r \)

Conversely

Let \( (x_i \otimes y_i) \) be subsequence in \( H_1 \otimes H_2 \).

Suppose \( A_1 = rI_1 \) and \( A_2 = -rI_2 \)

\[
\lim_{t \to \infty} (A_1 \otimes I_2 + I_1 \otimes A_2)(x_i \otimes y_i) = \lim_{t \to \infty} (rI_1 \otimes I_2 + I_1 \otimes -rI_2)(x_i \otimes y_i) = \lim_{t \to \infty} ((r-r)I_1 \otimes I_2)(x_i \otimes y_i) = 0
\]

Hence \( A_1 \otimes I_2 + I_1 \otimes A_2 \) is compact

We are now in a position to investigate the essential normality of operator tensor product

**Definition [3-1-6] [19]**

An operator \( A \) is called essential normal operator if \( A^*A - AA^* \) is compact.

The following theorem appeared in [19], and we give the details of the proof

**Theorem [3-1-7]**
Chapter three: Some properties of operators that are invariant under tensor product

\( A_1 \otimes A_2 \otimes K \otimes A_n \) is an essential normal operator if and only if one of the following holds:

1. \( A_i = 0 \) for some \( i \);
2. \( A_i \)'s are all normal (or compact);
3. \( A_i \) is essential normal for some \( i \) and \( A_j \)'s are all compactly normal for \( j \neq i \)

**Proof:**

By induction, we need only prove this theorem for the case \( n = 2 \)

1. Let \( A_i = 0 \) for some, \( i \). Suppose \( A_i = 0 \) let \( x_i \otimes y_i \) be a sequence in \( H_1 \otimes H_2 \)

\[
\lim_{t \to \infty} \left\| \left( A_1^* A_1 \otimes A_2^* A_2 - A_1 A_1^* \otimes A_2 A_2^* \right) x_i \otimes y_i \right\| = 0
\]

2. Suppose that \( A_1 \) and \( A_2 \) are normal then \( A_1 A_1^* = A_1^* A_1 \) and \( A_2 A_2^* = A_2^* A_2 \)

Let \( x_i \otimes y_i \) be a sequence in \( H_1 \otimes H_2 \)

\[
\lim_{t \to \infty} \left\| \left( A_1^* A_1 \otimes A_2^* A_2 - A_1 A_1^* \otimes A_2 A_2^* \right) x_i \otimes y_i \right\| = 0
\]

Hence \( A_1^* A_1 \otimes A_2^* A_2 - A_1 A_1^* \otimes A_2 A_2^* \) is compact

\( \Leftarrow \) Suppose \( A_1 \otimes A_2 \) is essential normal then by theorem (3-1-3) the proof is done

3. If \( A_i \) is essential normal then \( A_i^* A_i - A_i A_i^* \) is compact and \( A_2 \) is compactly normal i.e. \( A_2^* A_2 \) is compact.

\[
\left( A_i^* A_i - A_i A_i^* \right) \otimes A_2^* A_2 = A_i^* A_1 \otimes A_2^* A_2 - A_i A_i^* \otimes A_2 A_2^* \]

is compact
Chapter three

some properties of operators that are invariant under tensor product part II

$\Leftrightarrow$) Since $A_1 \otimes A_2$ is essential normal then $A_1^* A_1 \otimes A_2^* A_2 - A_1 A_1^* \otimes A_2 A_2^*$ is compact.

If $A_1$ is neither normal nor compact, then by theorem (3-1-3) \(\{A_2^* A_2, A_2 A_2^*\}\) are linearly dependent. $A_2$ is normal \((A_2^* A_2 - A_1 A_1^* ) \otimes A_2 A_2^*\) is compact then 

\([A_1]\) is compact. $A_1$ is essential normal operator.

$A_2^* A_2$ is compact. $A_2$ is compactly normal.

The following theorem appeared in [19], without proof, we give the proof.

**Theorem** [3-1-8]

$A_1 \otimes I_2 + I_1 \otimes A_2$ is essential normal if and only if $A_1$ and $A_2$ are normal.

**Proof:-**

Suppose $A_1 \otimes I_2 + I_1 \otimes A_2$ is essential normal

\[(A_1 \otimes I_2 + I_1 \otimes A_2)^* (A_1 \otimes I_2 + I_1 \otimes A_2) - (A_1 \otimes I_2 + I_1 \otimes A_2)\]

\[(A_1 \otimes I_2 + I_1 \otimes A_2)^*\] is compact

$A_1^* A_1 \otimes I_2 + A_1^* \otimes A_2 + A_1 \otimes A_2^* + I_1 \otimes A_2 A_2^* - A_1 A_1^* \otimes I_2 - I_1 \otimes A_2 A_2^* - A_1 \otimes A_2^* - A_1^* \otimes A_2$

is compact

\[ (A_1^* A_1 - A_1 A_1^*) \otimes I_2 + I_1 \otimes (A_2^* A_2 - A_2 A_2^*) \] is compact $A_1^* A_1 = A_1 A_1^*$, $A_1$ is normal and $A_2^* A_2 - A_2 A_2^*$ and $I_2$ is linearly independent then $A_2^* A_2 = A_2 A_2^*$, $A_2$ is normal.

$\Leftrightarrow$) Suppose that $A_1$ and $A_2$ are normal operators. Let $x_t \otimes y_t$ be a sequence in $H_1 \hat{\otimes} H_2$ then

\[
\lim_{t \to \infty} \left( (A_1^* A_1 - A_1 A_1^*) \otimes I_2 + I_1 \otimes (A_2^* A_2 - A_2 A_2^*) \right) x_t \otimes y_t = 0
\]

Hence $A_1 \otimes I_2 + I_1 \otimes A_2$ is essential normal operator.
**Chapter three**  some properties of operators that are invariant under tensor product part II

### 3.2. The relation between tensor product and elementary operators

**Definition [3-2-1] [22]**

Let $H$ be a complex separable Hilbert space, $B(H)$ the algebra of all bounded linear operators on $H$. For $1 \leq p \leq \infty$ the von Neumann-schatten class, $S - C^p_p(H)$ and $P \in \mathbb{R}$, is defined to be the set of all element $A$ in $B(H)$ such that

$$\sum_{k \in K} \|A\varphi_k, \varphi_k\|^p \leq \infty$$

for each orthonormal system $\{\varphi_k : k \in K\}$ in $H$.

If $p=1$ then the von Neumann –schatten class is known as Trace class, denoted by $S - C_1$

If $P=2$ then the von Neumann –schatten class is known as the Hilbert –Schmidt class, denoted by $S - C_2(H)$

**Definition [3-2-2] [20]**

Let $B(H)$ denoted the algebra of all bounded linear operators on an infinite dimensional complex separable Hilbert space $H$. For $A_1$ and $A_2$ in $B(H)$, Let: $\delta_{A_1,A_2}(X) : B(H) \rightarrow B(H)$ be defined by

$$\delta_{A_1,A_2}(X) = A_1 X - X A_2$$

for all $X \in B(H)$ \ldots (3-1)

Then $\delta_{A_1,A_2}(X)$ is called a generalized derivation. when $A_1 = A_2$, we simply write $\delta_{A_1}$ and is called a derivation.

It is clear that $\delta_{A_1,A_2}(X)$ is a linear map. In fact

$$\delta_{A_1,A_2}(\alpha X_1 + \beta X_2) = A_1 (\alpha X_1 + \beta X_2) - (\alpha X_1 + \beta X_2) A_2$$

$$= \alpha \delta_{A_1,A_2}(X_1) + \beta \delta_{A_1,A_2}(X_2)$$

and also is a bounded linear operator.
Chapter three

some properties of operators that are invariant under tensor product part II

And let $\tau_{A_1,A_2}(X) \colon B(H) \to B(H)$ be defined by

$$\tau_{A_1,A_2}(X) = A_1XA_2$$

for all $X \in B(H)$ \ldots (3-2)

It is clear that $\tau_{A_1,A_2}(X)$ is a linear map, in fact

$$\tau_{A_1,A_2}(\alpha X_1 + \beta X_2) = A_1(\alpha X_1 + \beta X_2)A_2 = A_1\alpha X_1A_2 + A_1\beta X_2A_2$$

$$= \alpha\tau_{A_1,A_2} + \beta\tau_{A_1,A_2}$$

We shall denote simply

$$( S - C_2 = S - C_2(H), \delta = \delta_{A_1,A_2} |_{S-C_2}, \tau = \tau_{A_1,A_2} |_{S-C_2} )$$

Now we give the definition of the Trace of an operator with some basic properties.

**Definition** [3-2-3]

Let $A$ be an operator in $S - C_1$, and $\{\varphi_k : k \in K\}$ is an orthonormal basis in $H$, then the trace of $A$, denoted by $tr(A)$, is defined by

$$tr(A) = \sum_{k \in K} \langle A\varphi_k, \varphi_k \rangle$$

And the definition does not depend on the choice of $\{\varphi_k\}$

The following remark appeared in [25]

**Remarks** [3-2-4]

Let $A_1, A_2$ be in $S - C_1, z \in \mathfrak{F}$, then

1. $tr(A^*) = \overline{\text{tr}(A_1)}$
2. $tr(zA_1) = z\text{tr}(A_1)$
3. $\text{tr}(A_1A_2) = r(A_1,A_2)$

**Proof:**
Chapter three  some properties of operators that are invariant under tensor product part II

1- \( \text{tr}(A_1^*) = \sum_{k \in K} \langle A_1^* \varphi_k, \varphi_k \rangle = \sum_{k \in K} \langle \varphi_k, A_1 \varphi_k \rangle = \sum_{k \in K} \langle A_1 \varphi_k, \varphi_k \rangle = \text{tr}(A_1) \)

2- \( \text{tr}(zA_1) = \sum_{k \in K} \langle zA_1 \varphi_k, \varphi_k \rangle = z \sum_{k \in K} \langle A_1 \varphi_k, \varphi_k \rangle = z \text{tr}(A_1) \)

3- \( \text{tr}(A_1A_2) = \sum_{k \in K} \langle A_1A_2 \varphi_k, \varphi_k \rangle = \sum_{k \in K} \langle \varphi_k, A_2^*A_1^* \varphi_k \rangle = \sum_{k \in K} \langle A_2^*A_1^* \varphi_k, \varphi_k \rangle \)

by (1 )

Hence \( \text{tr}(A_1A_2) = \text{tr}(A_2A_1) \).

Remark [3-2-5] [22]

It is easy to see that \( S - C_2 \) is a Hilbert space with respect to the inner product
\( \langle X, Y \rangle = \text{tr}(Y^*X) \quad X, Y \in S - C_2 \quad \ldots (3-3) \)

Definition [3-2-6] [28, p.7]

Let \( x \) and \( y \) be two given vectors in a Hilbert space \( H \), the symbol \( x \otimes y \) represents the operator on \( H \), defined by
\( (x \otimes y)f = \langle f, y \rangle x \) for all \( f \in H \).
Chapter three

some properties of operators that are invariant under tensor product part II

The following lemma is a simple consequence of the definition of $x \otimes y$

**Lemma**-[3-2-7]

1- $(x \otimes y)^* = (y \otimes x)$
2- $\lambda x \otimes y = \lambda (x \otimes y)$ for each complex number $\lambda$
3- $x \otimes (A y) = \overline{A}(x \otimes y)$
4- $(x_1 + x_2) \otimes y = (x_1 \otimes y) + (x_2 \otimes y)$
5- $x \otimes (y_1 + y_2) = x \otimes y_1 + x \otimes y_2$
6- $(x_1 \otimes y_1)(x_2 \otimes y_2) = (x_2, y_1)x_1 \otimes y_2$
7- $A(x \otimes y) = Ax \otimes y$ for every operator $A$ on $H$
8- $(x \otimes y)A = x \otimes A^*y$.

We study the relation between Tensor product operator and elementary operator

**Remark**[3-2-8] [15][3]

$\tau_{A_1,A_2}$ Can be identified with $A_1 \otimes A_2^*$

**Proof**:-

If $A_1 : K \to H$ and $A_2 : K' \to H'$ are operator $A_1 \otimes A_2 : K \otimes K' \to H \otimes H'$ so that if $x \in K$ and $x' \in K'$ $(A_1 \otimes A_2)(x \otimes y) = A_1 x \otimes A_2 x'$ consider the relation between the Hilbert–Schmidt operator $T : K' \to K$ and $S : H' \to H$ $y \in K, y' \in K'$

$T = y \otimes y'$ $S = A_1 y \otimes A_2 y'$

If, then

$A_1(y \otimes A_2 y')f' = \langle f', A_2 y' \rangle A_1 y = A_1[A_2^* f', y'] y]
= A_1[(y \otimes y') A_2^* f']$

$= (A_1TA_2^*) f'$

Hence $S = A_1TA_2^*$

63
We need the following theorem; we state it without proof Bram-Halmose theorem

**Theorem (3-2-9)**[2]

An operator $A$ on a Hilbert space $H$ is subnormal if and only if
\[
\sum_{n,m=0}^{r} \langle A^n f_m, A^m f_n \rangle \geq 0 \text{ for every finite set } f_0, f_1, \ldots, f_r \text{ in } H \text{ and there exists a positive constant } C \text{ such that }
\]
\[
\sum_{n,m=0}^{r} \langle A^{n+1} f_m, A^{m+1} f_n \rangle \leq C \sum_{m,n=0}^{r} \langle A^n f_m, A^m f_n \rangle \quad \ldots \ (3-4)
\]
For every finite set $f_0, f_1, \ldots, f_r$ in $H$

**Theorem (3-2-10)**

Let $\delta$ and $\tau$ be defined on $S - C_2$ by (3-1) and (3-2) then $\delta$ is subnormal if and only if $A_1$ and $A_2^*$ are subnormal operators.

Moreover, If $A_1 \neq 0$ and $A_2 \neq 0$ the same statement holds for $\tau, A_1, A_2$

**Proof:-**

Suppose first that $A_1$ and $A_2^*$ are subnormal operator and denoted by $M$ and $N^*$ their (not necessarily) normal extensions. We may assume that $M$ and $N$ act on the same Hilbert space $K \supset H$. Relative to the decomposition $K = H \oplus H^\perp$ the operators $M$ and $N^*$ can be represented by the matrices

\[
M = \begin{bmatrix} A_1 & A_1' \\ 0 & A_1'' \end{bmatrix}, \quad N^* = \begin{bmatrix} A_2^* & A_2' \\ 0 & A_2'' \end{bmatrix}
\]

Where $A_1', A_1'', A_2', A_2''$ are certain bounded operators.
Chapter three

Some properties of operators that are invariant under tensor product part II

Let $X \rightarrow \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix}$, $X \in S - C_2$

$\delta_{M,N}|_{S-C_2}$ defined on $S - C_2(K)$ by $\delta_{M,N} = MX - XN$ and $\delta_{M,N}|_{S-C_2}$ is normal operator.

To prove the converse

By using theorem (3-2-9)

Suppose that $\delta$ is subnormal. In order to apply (3-4) with $\delta$ inserted of $A_i$ express the powers $\delta^j$ by:

$$\delta^j X = \sum_{S=0}^{j} (-1)^S \binom{J}{S} A_1^{j-S} X A_2^S X \in S - C_2$$

Taking into account also the definition (3-3) of inner product in $S - C_2$, we see that (3-4) assume the form.

$$\sum_{J,k=0}^{n} \sum_{s=0}^{j} (-1)^s \binom{J}{S} A_1^{j-s} X_k A_2^s, \sum_{r=0}^{k} (-1)^r \binom{k}{r} A_1^{k-r} X_j A_2^r \geq 0 \quad (3-5)$$

$$\sum_{J,k=0}^{n} \sum_{s=0}^{j} \sum_{r=0}^{k} (-1)^{s+r} \binom{J}{S} \binom{k}{r} r A_2^s X_j A_1^{k-r} A_1^{j-r} X_k A_2^r \geq 0 \quad (3-5)$$

Where $X_1, X_2, K, X_n$ are arbitrary elements of $S - C_2$. Now let $f_j, g_j$

$J = 12, K, n$ be any vectors in $H$ and put $X_j = f_j \otimes g_j$ from (3-5)

$$\sum_{J,k=0}^{n} \sum_{s=0}^{j} \sum_{r=0}^{k} (-1)^s \binom{J}{S} A_1^{j-s} (f_k \otimes g_k) A_2^s, (-1)^r \binom{k}{r} A_1^{k-r} (f_j \otimes g_j) A_2^r \geq 0$$

$$\sum_{J,k=0}^{n} \sum_{s=0}^{j} \sum_{r=0}^{k} (-1)^{s+r} \binom{J}{S} \binom{k}{r} r A_2^s f_j \otimes A_1^{k-r} A_1^{j-r} A_2^r \geq 0 \quad (3-6)$$

Without loss of generality we may assume that $0$ is an approximate a eigen value of $A_2^*$ (otherwise we can replace $A_i$ and $A_2^*$ with $A_i - \alpha$ and
$A_2 - \alpha$ respectively, where $\bar{\alpha}$ is an approximate eigenvalue $A^*_2$; this is possible since $\delta_{A_1,A_2} = \delta_{A_1-\alpha,A_2-\alpha}$.

Let $(h_m)$ be the corresponding sequence of approximate eigenvectors that is, $\|h_m\| = 1$ and $\lim \|A^*_2 h_m\| = 0$ for fixed $m$. Put $g_1 = g_2 = K = g_n = h_m$ in (3-6), let $m$ tend to infinity. It follows that

$$\sum_{j,k=0}^{n} \langle A^{j'}_1 f_k, A^{k'}_1 f_j \rangle \geq 0$$

and this implies that $A_1$ is subnormal by the Bram-Halmos theorem.

The proof that subnormality of $\tau_{A_1,A_2}$ implies subnormality of $A_1$ and $A^*_2$ instead of (3-6) (derived in the same way as (3-6))

$$\sum_{j,k=0}^{n} \left\langle A^{j'}_1 f_k, A^{k'}_1 f_j, A^{j'}_2 g_j, A^{k'}_2 g_k \right\rangle \geq 0 \quad \text{...(3-7)}$$

since $A_1 \neq 0$ and $A_2 \neq 0$ then $\sigma(\tau_{A_1,A_2}) \neq 0$ by subnormality but $\sigma(\tau_{A_1,A_2}) = \sigma(A_1) \cdot \sigma(A_2)$, hence there is a $\beta \neq 0$ in the boundary of $\sigma(A^*_2)$, then $\beta$ is an approximate eigenvalue of $A^*_2$; let $h_m$ corresponding sequence of eigenvectors. Replace all $g_j$, $j = 1, 2, K, n$ in (3-7) with the same vector $h_m$ and then take $m \to \infty$ it follows

$$\sum_{j,k=0}^{n} \beta^k \beta^j \langle A^{j'}_1 f_k, A^{k'}_2 f_j \rangle \geq 0$$

and this implies $A_1$ is subnormal since $f_j$ are arbitrary and $\beta \neq 0$. The proof that $A^*_2$ is subnormal is similar.

**Remark [3-2-11][20]**

There exist some kind of characterization can not hold for general elementary operators. For example the operator $X \to A_1 X A_2 + A^*_1 X A^*_2$ is self-adjoint on $S = C_2$ for arbitrary $A_1, A_2 \in B(H)$.
In this section we define strongly stable operators and study a relation between strongly stable operators and tensor product of operators.

**Definition [3-3-1] [10]**

An operator $A$ is strongly stable if $\|A^n x\| \to 0$ as $n \to \infty$ for all $x \in H$.

**Definition [3-3-2] [10]**

An operator $A$ is power bounded if there exists a scalar $M \geq 0$ such that $\sup_n \|A^n\| \leq M$.

The following Remark appeared in [10].

**Remark [3-3-3]**

Every strongly stable is power bounded.

**Proof:** It is clear.

**Remark [3-3-4]**

There exists an operator $S$ and constant $c \geq 0$ such that $\|A^n x\| \geq c, \langle Sx, x \rangle \geq 0$ for all $x \in H$. $\ker S = \{ y \in H : A^n y \to 0 \text{ as } n \to \infty \}$ and $A^* S A = S$.

**Proposition [3-3-5]**

The power bounded operator $A$ is strongly stable if and only if (Positive) solution $X \geq 0$ of $A^* X A = X$ is $X = 0$.

**Proof:**

By induction it is easily seen that $A^* A^n X = X$.
\[ \langle X(x), x \rangle = \lim_{n \to \infty} \langle X(x), x \rangle = \lim_{n \to \infty} \left\langle (A^*)^n X A^* x, x \right\rangle = \lim_{n \to \infty} \langle X A^n x, A^n x \rangle = \lim_{n \to \infty} \langle X A^n x, A^n x \rangle \]
\[ \leq \| X \| \lim_{n \to \infty} \| A^n x \|^2 = 0 \]

\( \iff \) Suppose that the only solution \( X \geq 0 \) of \( A^* X A = X \) is \( X = 0 \), but there exists a non-trivial \( x \in H \) such that \( \| A^n x \| \to 0 \) \( n \to \infty \) by Remark (3-3-4) this is contradiction.

**Definition [3-3-6]**

An operator \( A \in B(H) \) is said to be uniformly stable if \( \| A^n \| \to 0 \) as \( n \to \infty \).

**Remark [3-3-7]**

Every uniform stable is strong stable

We state this remark without proof

**Remarks [3-3-8] [10]**

1. \( A \in B(H) \) is uniformly stable then \( r(A) \iota \) and \( A \) is similar to astric contraction

2. \( A \in B(H) \) is uniformly stable if and only if there exists an \( X \rangle \rangle 0 \) and a scalar \( \alpha \langle \iota \) such that \( A^* X A \leq \alpha X \)

In general the operator \( T = A_1 \otimes A_2 \) on \( H_1 \otimes H_2 \) uniformly stable if there exists an operator \( Q = Q_1 \otimes Q_2 \) and a scalar \( \alpha \iota \) such that \( T^* QT \leq \alpha Q \).
Chapter three 

some properties of operators that are invariant under tensor product part II

Theorem [3-3-9] [10]

a- Let $A_1$ and $A_2$ be power bounded operators on a separable Hilbert space $H$, then $A_1 \otimes A_2$ is strongly stable if and only if at least one of $A_1$ and $A_2$ is strongly stable.

b- $A_1 \otimes A_2$ is uniformly stable if and only if $A$ and $B$ are, where $A = cA_1$ and $B = c^{-1}A_2$ for some scalar $c \geq 0$

Proof:-

a- If $A_1$ and $A_2$ are strongly stable. $\exists \ X_i \geq 0 \ i=1,2$ such that

$$A_1^*X_1A_1 = X_1 \quad X_1 = 0$$
$$A_2^*X_2A_2 = X_2 \quad X_2 = 0$$

$$(A_1 \otimes A_2)^*(X_1 \otimes X_2)(A_1 \otimes A_2) = A_1^*X_1A_1 \otimes A_2^*X_2A_2 = X_1 \otimes X_2$$

Hence $A_1 \otimes A_2$ is strongly stable.

Conversely: - suppose that $A_1 \otimes A_2$ is strongly stable.

$$(A_1 \otimes A_2)^*(X_1 \otimes X_2)(A_1 \otimes A_2) = A_1^*X_1A_1 \otimes A_2^*X_2A_2 = X_1 \otimes X_2$$

$X_1 \otimes X_2 = 0$

By theorem (2-2-1) $\exists \ c \geq 0$ such that

$$A_1^*X_1A_1 = cX_1 \quad \text{and} \quad A_2^*X_2A_2 = c^{-1}X_2$$

Since $A_1$ and $A_2$ are power- bounded then

$$\sup_n \|A_1^n\| \leq M_1 \quad \text{and} \quad \sup_n \|A_2^n\| \leq M_2$$

$$c^n \|X_1\| = \|A_1^nX_1A_1^n\| < M_1^2 \|X_1\|$$

Then

$$c^n \|X_1\| \leq M_1^2 \|X_1\|$$

$$c^{-n} \|X_2\| = \|A_2^{-n}X_2A_2^{-n}\| < M_2^2 \|X_2\|$$

$$c^{-n} \|X_2\| \leq M_2^2 \|X_2\|$$

This impels that $c = 1$ and hence $A_1^*X_1A_1 = X_1$ and $A_2^*X_2A_2 = X_2$
b-If \( A_1 \) and \( A_2 \) are uniformly stable, then
\[
\| A_1^n \| \to 0 \quad \text{and} \quad \| A_2^n \| \to 0 \quad \text{as} \quad n \to \infty \quad m \to \infty
\]
\[
\lim_{n \to \infty} \lim_{m \to \infty} \left( A_1 \otimes A_2 \right)^{nm} = \left( A_1^n \right)^m \otimes \left( A_2^m \right)^n = \left( A_1^n \right)^m \left\| \left( A_2^m \right)^n \right\| = 0
\]

Then,
\[
r\left( A_1 \otimes A_2 \right) \lim_{n \to \infty} \left( A_1^n \otimes A_2^n \right)^{\frac{1}{n}} = \lim_{n \to \infty} \left( A \otimes B \right)^{\frac{1}{n}} = \lim_{n \to \infty} \left\| A^n \right\| \left\| B^n \right\|^{\frac{1}{n}}
\]

And hence \( A_1 \otimes A_2 \) is uniformly stable.

Conversely, If \( A_1 \otimes A_2 \) is uniformly stable, then

\[
A_1^* Q_1 A_1 \otimes A_2^* Q_2 A_2 \leq \alpha \left( Q_1 \otimes Q_2 \right)
\]

For some \( 0 < \alpha < 1 \) and \( Q_1 \otimes Q_2 >> 0 \) since \( Q_1 \otimes Q_2 \) is invertible if and only if \( Q_1 \) and \( Q_2 \) are. there exists non-zero scalar \( d \) such that \( X_1 = d Q_1 \) and \( X_2 = d^{-1} Q_2 \) are positive >> 0, the operators \( A_1^* X_1 A_1 \) and \( A_2^* X_2 A_2 \) being positive, there exists a scalar \( c > 0 \) such that

\[
A_1^* X_1 A_1 \leq c^2 \sqrt{\alpha} X_1 \quad \text{and} \quad A_2^* X_2 A_2 \leq c^{-2} \sqrt{\alpha} X_2
\]

This implies that \( A_1 \) and \( A_2 \) are uniformly stable.
References


4-Campbell, S.”Linear operators for which $T^*TT$ commute” Proc.Amer.Math.Soc. V.34 NO.1 (1972) 177-180.


الجداء التنسيوري للمؤثرات
على فضاءات هلبرت

رسالة مقدمة إلى
مجلس كلية العلوم - جامعة بغداد وهي
جزء من متطلبات نيل درجة الماجستير
في علوم الرياضيات

من قبل الطالبة
ميساء ماجد عبد المنعم التميمي

إيار 2005